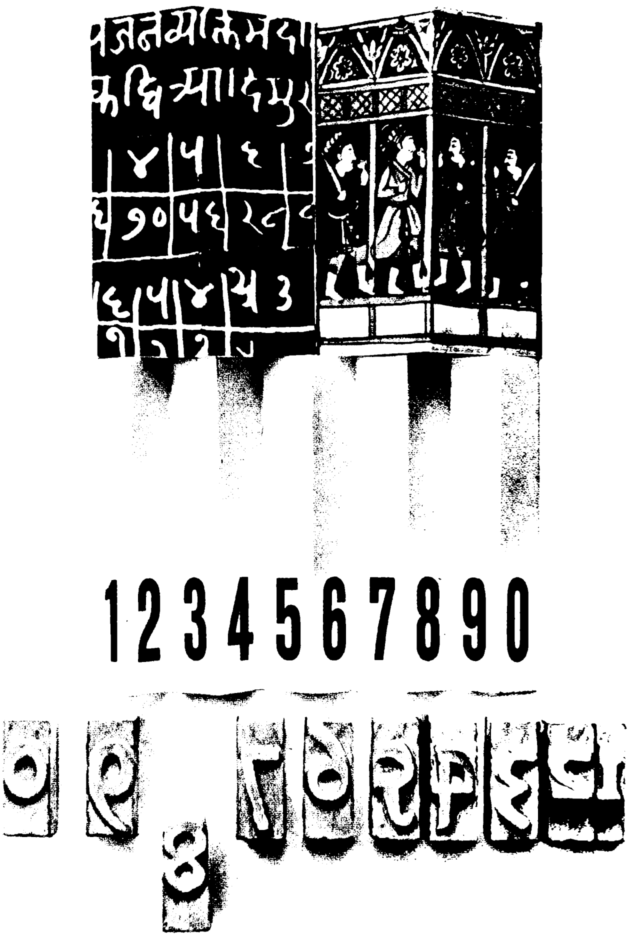
Smirnov: Problems on the equations of mathematical physics



* NOORDHOFF SERIES OF MONOGRAPHS AND TEXTBOOKS ON PURE & APPLIED MATHEMATICS

M. M. Smirnov

PROBLEMS ON THE EQUATIONS OF MATHEMATICAL PHYSICS

The collection is divided in three paragraphs. The first paragraph contains introductory exercises on the reduction of partial differential equations in canonical form. The second paragraph deals mainly with problems, the general solution of which can be formed by means of the method characteristics e.g. Cauchy's (or also Goursat's) and mixed problems. The third paragraph presents the method of separation of variables.

Contents:

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Problems on The Equations of Mathematical Physics

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PREFACE

The aim of the present collection of problems is to illustrate the theory of partial differential equations as it is given in various textbooks.

The problems of this collection are divided in three paragraphs. The first paragraph contains introductory excercizes on the reduction of partial differential equations to canonical form. The second paragraph deals mainly with problems, the general solution of which can be formed by means of the method of characteristics e.g. Cauchy's (or also Goursat's) and mixed problems.

In the third paragraph the most important method is presented, namely the separation of variables. This is done for mixed problems (for hyperbolic and parabolic equations) and for boundary value problems (elliptic equations).

The solutions of all excercizes are given. Most of the problems are accompanied by an explanation of the solution method used: so that this problem book can also be used for self study.

In preparing the present book the following books were consulted.

(A. N. Tychonoff and A. A. Samarskii – Equations of mathematical physics. Pergamon Press.).

(N. M. Günter and R. O. Kusmin – A collection of problems in higher mathematics).

(N. S. Koschkjakow – Important equations in mathematical physics) and other sources. Part of the problems is taken from the cited titles.

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Part I

PROBLEMS

§ 1. Reduction of partial differential equations with two independent variables to canonical form

The partial differential equation

$$a(x, y)\,\frac{\partial^2 u}{\partial x^2} + 2b(x, y)\,\frac{\partial^2 u}{\partial x \partial y} + c(x, y)\,\frac{\partial^2 u}{\partial y^2} + {}$$
$$+ F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \qquad (1)$$

is called an equation of hyperbolic type if $b^2 - ac > 0$, of parabolic type if $b^2 - ac = 0$ and of elliptic type if $b^2 - ac < 0$. Here a, b and c are functions of x and y twice continuously differentiable with respect to both variables.

To reduce equation (1) to canonical form one must write down the equation for the characteristic curves

$$a\,dy^2 - 2b\,dx\,dy + c\,dx^2 = 0, \qquad (2)$$

which breaks up into two equations

$$a\,dy - (b + \sqrt{b^2 - ac})\,dx = 0, \qquad (3)$$

$$a\,dy - (b - \sqrt{b^2 - ac})\,dx = 0 \qquad (4)$$

and find their general integrals.

1. Equations of Hyperbolic Type

$$b^2 - ac > 0.$$

The general integrals $\varphi(x, y) = c_1$, and $\psi(x, y) = c_2$ of equations (3) and (4) are real and different and define two different families of real characteristic curves.

Transformation to new variables (ξ, η) instead of (x, y)

$$\xi = \varphi(x, y)$$
$$\eta = \psi(x, y)$$

reduces equation (1) to:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = F_1\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right). \tag{5}$$

This is the canonical form of the partial differential equations of hyperbolic type.

2. Equations of parabolic Type

$$b^2 - ac = 0.$$

Equations (3) and (4) coincide and we obtain one general integral of equation (2): $\varphi(x, y) = c$.
In this case let us put $\xi = \varphi(x, y)$ and $\eta = \eta(x, y)$ where the Jacobian $\dfrac{D(\xi, \eta)}{D(x, y)}$ does not vanish in the domain of parabolicity.
Equation (1) reduces to

$$\frac{\partial^2 u}{\partial \eta^2} = F_2\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right). \tag{6}$$

This is the canonical form of the equations of parabolic type.

3. Equations of Elliptic Type

$$b^2 - ac < 0.$$

The general integrals of equations (3) and (4) are complex conjugated: they define two families of complex characteristic curves. The general integral of equation (3) has the form

$$\varphi(x, y) + i\psi(x, y) = c,$$

where $\varphi(x, y)$ and $\psi(x, y)$ are real functions.

With

$$\xi = \varphi(x, y)$$
$$\eta = \psi(x, y)$$

equation (1) reduces to

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = F_3\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right). \tag{7}$$

This is the canonical form of the equations of elliptic type.
In case of ellipticity we suppose that the coefficients a, b and c are analytic functions.
We remark that a transformation of the derivatives with respect to x and y to the new variables ξ and η is represented by the formulas:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y};$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\cdot\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial x} + \\
&\quad + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial^2 \eta}{\partial x^2}; \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\cdot\frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial y} + \\
&\quad + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial^2 \eta}{\partial y^2}; \\
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial^2 u}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial x}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial x}\right) + \\
&\quad + \frac{\partial^2 u}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial x}\cdot\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial^2 \eta}{\partial x \partial y}.
\end{aligned} \tag{8}$$

More details can be found in:
(I. G. Petrowski: Lectures on partial differential equations. Gostekhizdat 1953)

Example: The equation

$$x^2 \frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} = 0. \tag{9}$$

is of hyperbolic type because

$$b^2 - ac = x^2y^2 > 0 \text{ (for } x \neq 0 \text{ and } y \neq 0).$$

Following the general theory we consider the characteristic equation

$$x^2dy^2 - y^2dx^2 = 0$$

or

$$xdy + ydx = 0, \; xdy - ydx = 0.$$

After integration of these equations we obtain

$$xy = c_1 \text{ and } \frac{y}{x} = c_2.$$

We transform to new variables ξ and η:

$$\xi = xy \text{ and } \eta = \frac{y}{x}.$$

Formula 8 gives:

$$\frac{\partial^2 u}{\partial x^2} = y^2 \frac{\partial^2 u}{\partial \xi^2} - 2\frac{y^2}{x^2}\cdot\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{y^2}{x^4}\cdot\frac{\partial^2 u}{\partial \eta^2} + 2\frac{y}{x^3}\cdot\frac{\partial u}{\partial \eta};$$

$$\frac{\partial^2 u}{\partial y^2} = x^2 \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{1}{x^2}\cdot\frac{\partial^2 u}{\partial \eta^2}.$$

Substitution of these values in (9), reduces this equation to canonical form:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{1}{2\xi}\cdot\frac{\partial u}{\partial \eta} = 0.$$

Reduce the following equations to canonical form:

1. $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} - 3\dfrac{\partial^2 u}{\partial y^2} + 2\dfrac{\partial u}{\partial x} + 6\dfrac{\partial u}{\partial y} = 0.$

2. $\dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x \partial y} + 5\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0.$

3. $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} + \alpha\dfrac{\partial u}{\partial x} + \beta\dfrac{\partial u}{\partial y} + cu = 0.$

4. $$\frac{\partial^2 u}{\partial x^2} - 2\cos x \frac{\partial^2 u}{\partial x \partial y} - (3 + \sin^2 x) \frac{\partial^2 u}{\partial y^2} - y \frac{\partial u}{\partial y} = 0.$$

5. $$y^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + 2x^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} = 0.$$

6. $$\operatorname{tg}^2 x \frac{\partial^2 u}{\partial x^2} - 2y \operatorname{tg} x \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + \operatorname{tg}^3 x \frac{\partial u}{\partial x} = 0.$$

7. $$y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

8. $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} - 3y^2 \frac{\partial^2 u}{\partial y^2} - 2x \frac{\partial u}{\partial x} + {} + 4y \frac{\partial u}{\partial y} + 16x^4 u = 0.$$

9. $$(1 + x^2) \frac{\partial^2 u}{\partial x^2} + (1 + y^2) \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0.$$

10. $$\sin^2 x \frac{\partial^2 u}{\partial x^2} - 2y \sin x \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

11. $$\frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y^2} + \alpha \frac{\partial u}{\partial y} = 0 \quad (\alpha = \text{const.}).$$

§ 2. The Method of Characteristics

Find general solutions of the following equations:

12. $$\frac{\partial^2 u}{\partial x^2} - 2\sin x \frac{\partial^2 u}{\partial x \partial y} - \cos^2 x \frac{\partial^2 u}{\partial y^2} - \cos x \frac{\partial u}{\partial y} = 0.$$

13. $$x^2 \frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0.$$

14. $$\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) = x^2 \frac{\partial^2 u}{\partial y^2}.$$

15. $(x-y)\dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0.$

16. $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + 2yz \frac{\partial^2 u}{\partial y \partial z} + z^2 \frac{\partial^2 u}{\partial z^2} + 2zx \frac{\partial^2 u}{\partial z \partial x} = 0.$$

17. $$\frac{\partial^2 u}{\partial t^2} = a_{11} \frac{\partial^2 u}{\partial x^2} + 2a_{12} \frac{\partial^2 u}{\partial x \partial y} + a_{22} \frac{\partial^2 u}{\partial y^2} \quad (a_{11}a_{22} = a_{12}^2),$$

a_{11}, a_{12} and a_{22} are positive and constant.

18. $$\frac{\partial^4 u}{\partial x^4} - 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0.$$

19. Find the regions where the equation

$$(1-x^2)\frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} - (1+y^2) \frac{\partial^2 u}{\partial y^2} - 2x \frac{\partial u}{\partial x} - 2y \frac{\partial u}{\partial y} = 0.$$

is hyperbolic, parabolic or elliptic and find its general solution.

20. Give necessary and sufficient conditions for the existence of functional invariant solutions of the following equation with constant coefficients. Find also its general solution.

$$L(u) = a_{11} \frac{\partial^2 u}{\partial x^2} + 2a_{12} \frac{\partial^2 u}{\partial x \partial y} + a_{22} \frac{\partial^2 u}{\partial y^2} + b_1 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} = 0 \qquad (*)$$

$(\delta = a_{12}^2 - a_{11}a_{22} > 0),$

We call a function u a functional invariant solution of (*), if for every (smooth enough) function F also $F(u)$ is a solution of the equation.

21. Show that if

$$a_{11}b_2^2 - 2a_{12}b_1b_2 + a_{22}b_1^2 + 4\delta c = 0,$$

then the equation with constant coefficients

$$L(u) + cu = 0 \tag{**}$$

has a general solution of the form:

$$u(x, y) = e^{(kx+my)/2\delta}[\psi_1(\alpha_1 x - y) + \psi_2(\alpha_2 x - y)];$$

Where ψ_1 and ψ_2 are arbitrary functions, $k = a_{22}b_1 - a_{12}b_2$, $m = a_{11}b_2 - a_{12}b_1$ and α_1, α_2 are the roots of the equation

$$a_{11}\alpha^2 - 2a_{12}\alpha + a_{22} = 0.$$

If the condition given in this problem is not fulfilled, reduce then equation (**) to the form

$$\frac{\partial^2 v}{\partial\xi\partial\eta} = \bar{c}v,$$

with

$$\bar{c} = \frac{a_{11}^2(\alpha_1 b_1 - b_2)(\alpha_2 b_1 - b_2) + 4a_{11}c\delta}{16\delta^2,},$$

$$\xi = \alpha_1 x - y, \qquad \eta = \alpha_2 x - y.$$

22. Show that a general solution of the equation:

$$\frac{\partial^2 v}{\partial x \partial y} = v$$

has the form:

$$v(x, y) = \int_0^x \psi_1(t) J_0(2i\sqrt{y(x - t)}\,dt +$$
$$+ \int_0^y \psi_2(t) J_0(2i\sqrt{x(y - t)}\,dt + v(0, 0) J_0(2i\sqrt{xy})$$

where $\psi_1(t)$ and $\psi_2(t)$ are arbitrary functions.

23. Show that a general solution of the equation

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{n}{x - y}\cdot\frac{\partial u}{\partial x} + \frac{m}{x - y}\cdot\frac{\partial u}{\partial y} = 0$$

has the form

$$u(x, y) = \frac{\partial^{m+n-2}}{\partial x^{m-1} \partial y^{n-1}} \left[\frac{X(x) - Y(y)}{x - y} \right]$$

where $X(x)$ and $Y(y)$ are arbitrary functions.

24. Find the general solution of the equation

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{2}{x-y} \cdot \frac{\partial u}{\partial x} + \frac{3}{x-y} \cdot \frac{\partial u}{\partial y} - \frac{3}{(x-y)^2} u = 0.$$

25. Show that a general solution of the equation:

$$E(\alpha, \beta) = \frac{\partial^2 u}{\partial x \partial y} - \frac{\beta}{x-y} \cdot \frac{\partial u}{\partial x} + \frac{\alpha}{x-y} \cdot \frac{\partial u}{\partial y} = 0$$

$$(0 < \alpha, \beta < 1, \alpha + \beta \neq 1)$$

has the form:

$$u(x, y) = (y - x)^{1-\alpha-\beta} \int_0^1 \varphi[x + (y - x)t] t^{-\alpha} (1 - t)^{-\beta} dt +$$

$$+ \int_0^1 \psi[x + (y - x)t] t^{\beta-1} (1 - t)^{\alpha-1} dt$$

where φ and ψ are arbitrary functions.

26. Show that a general solution of the equation:

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{n}{x-y} \cdot \frac{\partial u}{\partial x} - \frac{m}{x-y} \cdot \frac{\partial u}{\partial y} = 0$$

has the form:

$$u(x, y) = (x - y)^{m+n+1} \frac{\partial^{m+n}}{\partial x^n \partial y^m} \left[\frac{X(x) - Y(y)}{x - y} \right]$$

where $X(x)$ and $Y(y)$ are arbitrary functions.

27. Find a general solution of the equation:

$$\frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y^2} + \alpha \frac{\partial u}{\partial y} = 0 \qquad (\tfrac{1}{2} < \alpha < 1, y < 0).$$

28. Find propagating waves for the equation:

a) $\dfrac{\partial^2 u}{\partial t^2} = x^2 \dfrac{\partial^2 u}{\partial x^2} + \tfrac{1}{4} u,$

b) $$\frac{\partial^2 u}{\partial x^2} - 4x^2 \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial t} + u = 0$$

29. Find a singular solution for each of the equations

a) $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + cu,$$

b) $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + cu,$$

which vanishes on the surface of a characteristic cone in infinity.

30. Find the solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} = 0,$$

satisfying the initial conditions

$$u|_{y=0} = 3x^2, \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0.$$

31. Calculate the solution of the equation

$$(1 + x^2)\frac{\partial^2 u}{\partial x^2} - (1 + y^2)\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$$

with initial conditions.

$$u|_{y=0} = \psi_0(x), \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = \varphi_1(x).$$

32. Find the solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + 2\cos x\frac{\partial^2 u}{\partial x \partial y} - \sin^2 x\frac{\partial^2 u}{\partial y^2} - \sin x\frac{\partial u}{\partial y} = 0$$

satisfying the initial conditions

$$u|_{y=\sin x} = \varphi_0(x), \quad \left.\frac{\partial u}{\partial y}\right|_{y=\sin x} = \varphi_1(x).$$

33. Solve the equation

$$x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} - 3y^2\frac{\partial^2 u}{\partial y^2} = 0$$

subject to the initial conditions

$$u|_{y=1} = \varphi_0(x), \quad \left.\frac{\partial u}{\partial y}\right|_{y=1} = \varphi_1(x).$$

34. Determine the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$$

which satisfies the initial conditions

$$u|_{t=0} = \varphi(r), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(r), \quad r = \sqrt{x^2 + y^2 + z^2},$$

where $\varphi(r)$ and $\psi(r)$ are given functions for $r \geqslant 0$ (spherical symmetry).

35. Solve the equation

$$\frac{\partial^2 u}{\partial y^2} - y^2\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\frac{\partial u}{\partial x} = 0$$

subject to the initial conditions

$$u|_{y=0} = \varphi_0(x), \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = \varphi_1(x).$$

36. Show that the equation

$$\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\frac{\partial u}{\partial y} = 0 \quad (y < 0)$$

has a unique solution which satisfies the following conditions

$$u|_{y=0} = \tau(x), \quad \left|\left.\frac{\partial u}{\partial y}\right|_{y=0}\right| \leqslant \kappa,$$

where κ is a finite constant.

37. Find the solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial u}{\partial y} = 0 \quad (\tfrac{1}{2} < \alpha < 1, y < 0),$$

satisfying the following initial conditions

$$u|_{x=y} = \tau(x), \quad (-y)^{\alpha} \frac{\partial u}{\partial y}\bigg|_{y=0} = \nu(x).$$

38. Solve the equation

$$\frac{\partial^2 v}{\partial x \partial y} = v,$$

subject to the conditions

$$v|_{y=x} = \varphi(x), \quad \frac{\partial u}{\partial x}\bigg|_{y=x} = \omega_1(x), \quad \frac{\partial v}{\partial y}\bigg|_{y=x} = \omega_2(x),$$

$$\varphi'(x) = \omega_1(x) + \omega_2(x).$$

39. Determine the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = a_{11} \frac{\partial^2 u}{\partial x^2} + 2a_{12} \frac{\partial^2 u}{\partial x \partial y} + a_{22} \frac{\partial^2 u}{\partial y^2} \quad (a_{11}a_{22} = a_{12}^2),$$

which satisfies the initial conditions

$$u|_{t=0} = f(x, y), \quad \frac{\partial u}{\partial t}\bigg|_{t=0} = F(x, y).$$

Find also a solution for the special case

$$f(x, y) = x^2 + y^2 \text{ and } F(x, y) = 0.$$

40. Solve the equation

$$\frac{\partial^4 u}{\partial x^4} - 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0,$$

subject to the initial conditions

$$u|_{y=0} = \tau(x), \quad \frac{\partial u}{\partial y}\bigg|_{y=0} = \nu(x),$$

$$\frac{\partial^2 u}{\partial y^2}\bigg|_{y=0} = \nu_1(x), \quad \frac{\partial^3 u}{\partial y^3}\bigg|_{y=0} = \nu_2(x).$$

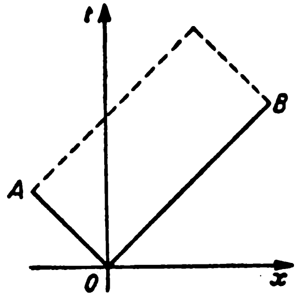

Fig. 1

41. A homogeneous string of linear density ρ, which is put under a large tension, slides over two fixed pulleys with constant velocity v. The part of the moving string between the pulleys executes transversal vibrations. Find the period of these vibrations.

42. Find the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

with prescribed values on two characteristic curves:
On the segment OA (fig. 1) of the characteristic curve $x + t = 0$ is

$$u(x, t) = \varphi(x);$$

on the segment OB of the characteristic curve $t - x = 0$ is

$$u(x, t) = \psi(x)$$

where $\varphi(0) = \psi(0)$.

43. Solve the equation

$$\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\frac{\partial u}{\partial y} = 0 \quad (y < 0),$$

if the values of u on the segment OB (fig. 2) of the characteristic curve $L_1: x - 2\sqrt{-y} = 0$ and on the segment AB of the characteristic curve $L_2: x + 2\sqrt{-y} = 1$ are given by

$$u(x, y)|_{L_1} = \varphi_1(x) \text{ for } 0 \leqslant x \leqslant \tfrac{1}{2},$$

$$u(x, y)|_{L_2} = \varphi_2(x) \text{ for } \tfrac{1}{2} \leqslant x \leqslant 1,$$

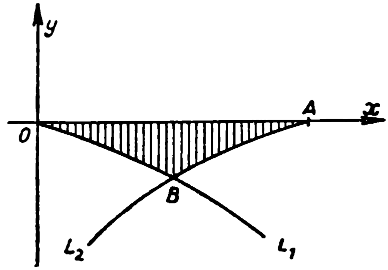

Fig. 2

with

$$\varphi_1(\tfrac{1}{2}) = \varphi_2(\tfrac{1}{2}).$$

44. Solve the equation

$$\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\frac{\partial u}{\partial y} = 0 \quad (y < 0),$$

if the values of u on the positive x-axis and the characteristic curve $L_1: x - 2\sqrt{-y} = 0$ (fig. 2) are given by

$$u(x, y)|_{y=0} = \varphi_1(x), \quad (x \geqslant 0)$$

$$u(x, y)|_{L_1} = \varphi_2(x)$$

where $\varphi_1(0) = \varphi_2(0)$.

45. Find the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

with given values of u on the segment OA of the characteristic curve $t - x = 0$ (fig. 3) and on the curve L. The curve L starts in the origin and is contained in the triangular region with sides wich are the characteristic curves $t \pm x = 0$ and L is such that it has exactly one intersection point with each characteristic curve $t - x = c$.

Discuss in particular the solution if L is a straight line

$$t - kx = 0 \quad (k > 0).$$

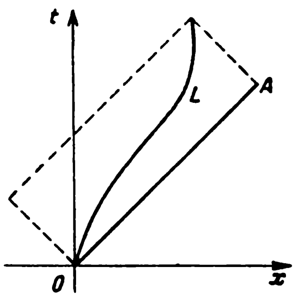

Fig. 3

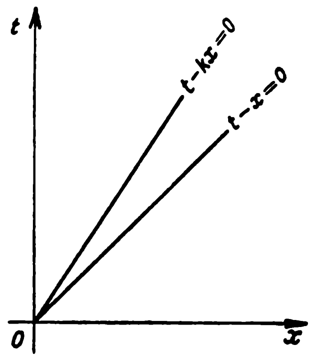

Fig. 4

46. Find the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

if the values of u and its normal derivatives are known on the positive x-axis, and if also the values of u on the semi-infinite straight line $t - kx = 0$ are given. $k > 1$ (fig. 4):

$$u|_{t=0} = \varphi_0(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \varphi_1(x) \quad (x \geqslant 0),$$

$$u|_{t=kx} = \psi(x) \quad (x \geqslant 0);$$

where

$$\varphi_0(0) = \psi(0).$$

Give conditions, such that the solution is smooth in the domain under consideration.

47. At time $t = 0$ a gas is contained inside a spherical volume of radius R, such that there the density is u_0, whereas it vanishes outside this volume. Find the density of the gas in a point M outside the volume at time $t > 0$.

48. A semi-infinite string ($x \geqslant 0$) of linear density ρ has a tension ρa^2 and is at rest. At time $t > 0$ the point $x = 0$ executes small vibrations $A \sin \omega t$. Show that the displacement of a point of the

string with abscis $x > 0$ is given by the formula:

$$u(x, t) = \begin{cases} 0 & \text{for } t < \dfrac{x}{a}, \\ A \sin \omega \left(t - \dfrac{x}{a}\right) & \text{for } t > \dfrac{x}{a}. \end{cases}$$

49. Derive the equation for longitudinal vibrations of a rod and integrate it, subject to the condition that one end is fixed and the other end is free.

50. A homgeneous string of length l is fixed at its ends. At time $t = 0$ the point $x = l/3$ is pulled aside over a small distance h and then released with initial velocity zero. Show that the motion of the string is described for $0 \leqslant t \leqslant l/3a$ by the formula

$$u(x, t) = \begin{cases} \dfrac{3h}{l} x & \text{for } 0 < x \leqslant \dfrac{l}{3} - at, \\ \dfrac{3h}{4l} x + \dfrac{9h}{4l}\left(\dfrac{l}{3} - at\right) & \text{for } \dfrac{l}{3} - at < x < \dfrac{l}{3} + at, \\ \dfrac{3h}{2l}(l - x) & \text{for } \dfrac{l}{3} + at < x \leqslant l \end{cases}$$

51. A semi-infinite tube ($x \geqslant 0$) filled with an ideal gas has at one end ($x = 0$) a freely moving piston of mass 1. At the moment $t = 0$, the piston is given an initial verlocity v_0 by the impact of a hammer. Describe the process of wave propagation in the gas if the initial displacements and the initial velocity of the gas are zero.

52. An infinite string having a concentrated mass M at the point $x = 0$ is at rest. At time $t = 0$ the mass M is given the initial velocity v_0. Prove that for $t > 0$ the vibrating string has the shape shown in fig. 5 where $u_1(x, t)$ is an ingoing wave determined by the formula

$$u_1(x, t) = \frac{Mav_0}{2T_0}[1 - e^{(2T_0/Ma^2)(x-at)}] \quad \text{for } x - at < 0,$$

$$u_1(x, t) = 0 \quad \text{for } x - at > 0,$$

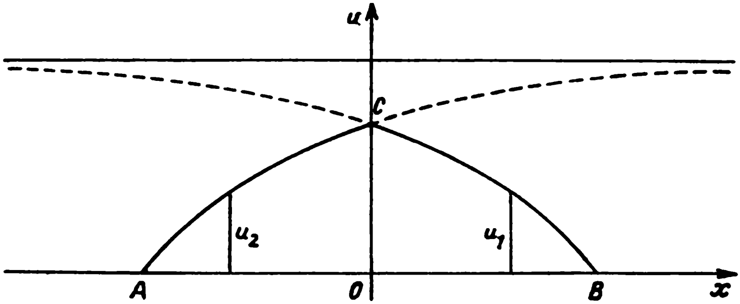

Fig. 5

and $u_2(x, t)$ is an outgoing wave determined by the formula

$$u_2(x, t) = \frac{Mav_0}{2T_0}[1 - e^{-(2T_0/Ma^2)(x+at)}] \quad \text{for } x + at > 0,$$

$$u_2(x, t) = 0 \quad \text{for } x + at < 0$$

T_0 is the tension of the string.

53. A rod of length l moving with velocity v_1 overtakes an identical rod moving in the same direction with velocity $v_2 < v_1$. Determine the distribution of the velocities of the longitudinal waves in the two rods. It is assumed that the impact is totally inelastic.

54. A mass M with velocity v strikes the upper free end ($x = l$) of a rod, the lower end of which ($x = 0$) is kept fixed. Find the velocity of the longitudinal displacement of that cross-section, which has at the moment $t = 4l/a$ the abscis $x = l/2$.

55. Solve the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = u,$$

subject to the initial conditions

$$u|_{t=0} = \varphi_0(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \varphi_1(x) \quad (0 \leqslant x \leqslant 2l)$$

and the boundary-conditions

$$u|_{x=0} = f_1(t), \quad u|_{x=2l} = f_2(t) \quad (t \geqslant 0).$$

56. Determine the solution of the system of equations

$$\frac{\partial u}{\partial x} = -b(u - v), \quad \frac{\partial v}{\partial y} = a(u - v),$$

satisfying the conditions.

$$u|_{x=0} = 0, \quad v|_{y=0} = 1.$$

57. Determine the solution of the system of the equations

$$\frac{\partial u}{\partial t} = -\rho a \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial t} = -\frac{a}{\rho} \frac{\partial u}{\partial x},$$

satisfying the initial conditions

$$u|_{t=0} = 0, \quad v|_{t=0} = 0 \quad (0 \leqslant x \leqslant l)$$

and the boundary conditions

$$u|_{x=0} = 0, \quad v|_{x=l} = \alpha(t)(1 + \beta u) - 1 \quad (t \geqslant 0)$$

where a, β and ρ are constant and $\alpha(t)$ is a given function.

§ 3. Separation of variables

We look for the solution of the equation

$$\rho(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[p(x) \frac{\partial u}{\partial x}\right] - q(x)u, \tag{1}$$

which satisfies the boundary conditions

$$\begin{aligned} \alpha u(0, t) + \beta \frac{\partial u(0, t)}{\partial x} &= 0, \\ \gamma u(l, t) + \delta \frac{\partial u(l, t)}{\partial x} &= 0 \end{aligned} \tag{2}$$

and the initial conditions

$$u(x, 0) = \varphi_0(x), \quad \frac{\partial u(x, 0)}{\partial t} = \varphi_1(x) \quad (0 \leqslant x \leqslant l) \tag{3}$$

The functions $g(x)$, $p(x)$ and $q(x)$ are sufficiently smooth and satisfy the conditions: $p(x) > 0$, $\rho(x) > 0$ and $q(x) \geqslant 0$.
Suppose $u(x, t)$ is a nontrivial solution of (1), which satisfies the boundary conditions (2) and can be represented in the form of a product

$$u(x, t) = T(t)X(x).$$

Substituting (4) into (1) we obtain

$$\rho(x)T''(t)X(x) = T(t)\frac{d}{dx}\left[p(x)\frac{dX}{dx}\right] - q(x)T(t)X(x)$$

or

$$\frac{\dfrac{d}{dx}\left[p(x)\dfrac{dX}{dx}\right] - q(x)X(x)}{\rho(x)X(x)} = \frac{T''(t)}{T(t)} = -\lambda,$$

where λ is a constant.
It follows that this is equivalent to

$$\frac{d}{dx}\left[p(x)\frac{dX}{dx}\right] + [\lambda\rho(x) - q(x)]X = 0, \tag{5}$$

$$T''(t) + \lambda T(t) = 0. \tag{6}$$

Since $T(t)$ does not vanish identically the function (4) satisfies the boundary conditions (2) if and only if $u(x, t)$ satisfies

$$\begin{aligned} \alpha X(0) + \beta X'(0) &= 0, \\ \gamma X(l) + \delta X'(l) &= 0 \end{aligned} \tag{7}$$

Thus we arrive for the evaluation of the function $X(x)$ at the following boundary value problem for an ordinary differential equation: Find those values λ, called eigenvalues, for which non-trivial solutions of (5) exist, which satisfy the boundary conditions (7) and find also the non-trivial solutions, called "eigenfunctions" associated with these.
The following theorems can be proven:

1. There exist an infinite number of eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$, with corresponding eigenfunctions $X_1(x)$, $X_2(x)\ldots$

2. For $q(x) \geqslant 0$ all the eigenvalues are positive.

3. Eigenfunctions associated with different eigenvalues are orthogonal and can be normalized with weight $\rho(x)$.

$$\int_0^l \rho(x) X_n(x) X_m(x)\, dx = \begin{cases} 0 \text{ for } m \neq n, \\ 1 \text{ for } m = n. \end{cases} \tag{8}$$

4. (Theorem of Steklov). Every function $f(x)$, which satisfies the boundary conditions (7) and which has a continuous first order and a piece-wise continuous second order derivative, can be expanded in an absolutely and uniformly convergent series with respect to the eigenfunctions $X_n(x)$.

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x), \quad c_n = \int_0^l \rho(x) X_n(x) f(x)\, dx.$$

Furthermore equation (6) has a solution for every eigenvalue λ_n. The general solution of (6) for $\lambda = \lambda_n$, which we shall denote by $T_n(t)$ has the form:

$$T_n(t) = A_n \cos \sqrt{\lambda_n} t + B_n \sin \sqrt{\lambda_n} t.$$

Where A_n and B_n are arbitrary constants.
We get an infinite set of solutions of equation (1) of the form:

$$u_n(x, t) = T_n(t) X_n(x) = (A_n \cos \sqrt{\lambda_n} t + B_n \sin \sqrt{\lambda_n} t) X_n(x).$$

In order to satisfy the initial conditions (3) we form the series

$$u(x, t) = \sum_{n=1}^{\infty} (A_n \cos \sqrt{\lambda_n} t + B_n \sin \sqrt{\lambda_n} t) X_n(x). \tag{9}$$

If this series and the series of the first order derivatives with respect to x and t converge uniformly, then its sum satisfies equation (1) and the boundary conditions (2).
The initial conditions (3) on $u(x, t)$ become

$$u(x, 0) = \sum_{n=1}^{\infty} A_n X_n(x) = \varphi_0(x), \tag{10}$$

$$\frac{\partial u(x, 0)}{\partial t} = \sum_{n=1}^{\infty} \sqrt{\lambda_n} B_n X_n(x) = \varphi_1(x). \tag{11}$$

If the series (10) and (11) converge uniformly we can calculate the

coefficients A_n and B_n by multiplying both sides of equations (10) and (11) by $\rho(x).X_n(x)$ and by integrating with respect to x from 0 to l.
With the aid of (8) we get

$$A_n = \int_0^l \rho(x)\varphi_0(x)X_n(x)\,dx,$$

$$B_n \frac{1}{\sqrt{\lambda_n}} \int_0^l \rho(x)\varphi_1(x)X_n(x)\,dx.$$

Substitution of these values of A_n and B_n in the series (9) gives the solution of our problem. (For more details see I. G. Petrowski, Lectures on partial differential equations).

1. Equations of hyperbolic type

58. A homogeneous string is fixed at the ends $x = 0$ and $x = l$ and has at time $t = 0$ the form

$$u(x, 0) = \frac{16}{5} h\left[\left(\frac{x}{l}\right)^4 - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)\right]$$

where h is positive and sufficiently small. The initial velocities are zero. Investigate the free vibrations of the string.

59. A homogeneous string with fixed ends $x = 0$ and $x = l$ has at time $t = 0$ the form of a parabola symmetric with respect to the normal in the point $x = l/2$. Find the deflection of the string from the equilibrium position if the initial velocities are zero.

60. A homogeneous string is fixed at $x = 0$ and $x = l$. The point $x = c$ of the string is displaced over a small distance h and released at time $t = 0$. The initial velocities are zero. Find the deflection $u(x, t)$ of the string for $t > 0$.

61. A homogeneous string spanned between two fixed points is in equilibrium. At time $t = 0$, it is excited by the impact of a hammer at the point $x = c$, so that it obtains at this point a constant velocity v_0. Find the deflection $u(x, t)$ of the string at time $t > 0$.

Investigate two cases:

a) The string is exited with the initial velocity

$$\frac{\partial u(x, 0)}{\partial t} = \begin{cases} v_0 \text{ for } |x - c| < \dfrac{\pi}{2h}, \\ 0 \text{ for } |x - c| > \dfrac{\pi}{2h} \end{cases}$$

This corresponds to the impact of a plane rigid hammer with width π/h at the point $x = c$.

b) The string is excited with the initial velocity

$$\frac{\partial u(x, 0)}{\partial t} = \begin{cases} v_0 \cos h(x - c) & \text{for } |x - c| < \dfrac{\pi}{2h}, \\ 0 & \text{for } |x - c| > \dfrac{\pi}{2h} \end{cases}$$

This case corresponds to the impact of a convex rigid hammer with width π/h at the point $x = {}^l/_2$.

62. One end ($x = 0$) of a string of length l is fixed, while the other end is attached to a ring, the mass of which can be neglected. The ring, which can move along a smooth rod, is displaced over a small distance from the equilibrium position and is released at timet $t = 0$. Describe the vibrations of the string for $t > 0$.

63. A tube, one end of which is open, moves in a direction parallel to its axis with constant velocity v_0 and stops instantaneously at time $t = 0$. Find the vibrations of the gas in the tube at a distance x of the closed end.

64. Integrate the equation for small longitudinal vibrations of a cylindrical rod, one end of which is rigidly fixed, while the other end is free.

65. One end of a rod is rigidly fixed and a force Q is applied to the other end. Find the longitudinal vibrations of the rod when the force Q instantaneously disappears at time $t = 0$.

66. Investigate the free longitudinal vibrations of a homogeneous rod of length l with free ends.

67. A force is applied to the ends of a homogeneous rod of lengtht l, such that it is compressed to a lengtht $2l(1 - \varepsilon)$. At time $t = 0$ the force disappears. Show that the displacement $u(x, t)$ of the cross-section with abscis x is given by the formula

$$u(x, t) = \frac{8\varepsilon l}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)^2} \sin \frac{(2n+1)\pi x}{2l} \cos \frac{(2n+1)\pi a t}{2l}$$

where $x = 0$ corresponds with the midpoint of the rod and where a is the velocity of the longitudinal wave.

68. Torsional vibrations of a rod are vibrations where the cross-sections are given small angular displacements in planes perpendicular to the axis. Derive the differential equation for small torsional vibrations of a homogeneous cylindrical rod and integrate this equation subject to the boundary condition that one end of the rod is rigidly fixed and that at the other end a disk is attached.

69. A homogeneous rod has lengtht l and cross-sectional area σ. One end ($x = 0$) is rigidly fixed and at the other end a concentrated mass M is attached. The rod is stretched by a force Q. Find the longitudinal vibrations if the force disappears instantaneously.

70. The ends of a homogeneous rod of lengtht l are constrained to move on two straight lines parallel to the u-axis by means of elastical forces. Investigate the free transversal vibrations of the rod when the initial displacements and velocities of all of its points are given.

71. Investigate the free vibrations of a string with fixed ends and which vibrates in the midpoint, if the resistance is proportional to the velocity.

72. Calculate the forced transversal vibrations of a string with one fixed end ($x = 0$), if at the end $x = l$ a harmonic force is applied, such that it moves according to the law $u(l, t) = A \sin \omega t$.

73. A rod of lengtht l with one fixed end ($x = 0$) is in equilibrium. At time $t = 0$ a force Q per unit cross-section directed along the axis of the rod is applied to the other free end. Find the longitudinal vibrations at time $t > 0$.

74. Find the longitudiual vibrations of a homogeneous cylindrical rod of lenght l if the end $x = 0$ of the rod is rigidly fixed and a force $F = A \sin \omega t$ is applied to the other end such that its direction coincides with the axis of the rod.

75. Investigate the forced vibrations of an inhomogeneous rod which consists of two homogeneous rods, joined together at the point $x = c$, if one end of the rod is rigidly fixed and the other end moves according to the law $u(l, t) = A \sin \omega t$.

76. A vertical rod is clamped in such a way that the displacement of each of its points is equal to zero. At time $t = 0$ the constraining forces are removed but the upper end is kept fixed. Find the forced vibrations of the rod.

77. A homogeneous string of lenght l with fixed ends vibrates under the influence of an external harmonic force $F(x, t) = \rho f(x) \cdot \sin \omega t$ (per unit length). Find the deflection $u(x, t)$ for arbitrary initial conditions. Investigate the possibility of resonance and find the solution in that case.

78. Solve the boundary value problem for the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + b \,\mathfrak{Sin}\, x$$

with initial and boundary conditions

$$u(0, t) = u(l, t) = 0; \quad u(x, 0) = 0; \quad \frac{\partial u(x, 0)}{\partial t} = 0.$$

79. Solve the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + bx(x - l)$$

with zero initial conditions and boundary conditions

$$u(0, t) = 0, \quad u(l, t) = 0.$$

80. Solve the equation

$$\frac{\partial^2 u}{\partial^2 x} - a^2 \frac{\partial^2 u}{\partial t^2} - 2h \frac{\partial u}{\partial t} - b^2 u = 0$$

with zero initial conditions and boundary conditions

$$u(0, t) = A, \quad u(l, t) = 0.$$

81. A homogeneous heavy string of lenght l, which is fixed at the point $x = l$ of the vertical axis, rotates with a constant angular verlocity ω about this axis. Derive the equation for small vibrations of the string and show that the deviation from the equilibrium position is given by the formula:

$$u(x, t) = \sum_{k=1}^{\infty} (A_k \cos a\lambda_k t + B_k \sin a\lambda_k t) \mathrm{J}_0\left(\mu_k \sqrt{\frac{x}{l}}\right)$$

with

$$A_k = \frac{1}{l\mathrm{J}_1^2(\mu_k)} \int_0^l f(x) \mathrm{J}_0\left(\mu_k \sqrt{\frac{x}{l}}\right) dx,$$

$$\lambda_k = \sqrt{\frac{\mu_k^2}{4l} - \left(\frac{\omega}{a}\right)^2},$$

$$B_k = \frac{1}{a\lambda_k l \mathrm{J}_1^2(\mu_k)} \int_0^l F(x) \mathrm{J}_0\left(\mu_k \sqrt{\frac{x}{l}}\right) dx;$$

where $\mu_1, \mu_2, \ldots$ are the positive zeros of the Besselfunction $\mathrm{J}_0(x)$.

82. Find the steady-state vibrations of a homogeneous circular membrane of radius R fixed around the edge, if the initial deflection is a paraboloid of revolution and the initial velocities equal zero.

83. Investigate the free radial vibrations of a membrane fixed around the edge and vibrating in the center, if the restistance is proportional to the velocity.

84. A heavy homogeneous string of lenght l fixed at the upper end $x = l$ is displaced from its equilibrium position and released with an initial velocity equal to zero. Show that the equation for the small vibrations of the string under the influence of the gravitational force is given by

$$\frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right) = \frac{1}{a^2} \cdot \frac{\partial^2 u}{\partial t^2},$$

where $a = \sqrt{g}$.

Integrate this equation subject to the conditions given in the problem.

85. Given a very long tube of radius R (which may be considered to be infinitely long at both ends). Investigate the small transverse vibrations of a homogeneous gas with which the tube is filled.

86. A flexible homogeneous string rotates with constant angular velocity ω about a vertical axis. At time $t = 0$ the relative equilibrium position is disturbed and its points may obtain an initial velocity. Find the deflection $u(x, t)$ of the string at time $t > 0$.

87. A homogeneous circular membrane of radius R fixed around the edge is in equilibrium, while a tension T is present. At time $t = 0$ a constant pressure P is applied to one side of the membrane. Show that the deflection of a point of the membrane is given by the formula:

$$u(r, t) = \frac{P}{T}\left[\tfrac{1}{4}(R^2 - r^2) - 2R^2 \sum_{k=1}^{\infty} \frac{J_0\left(\mu_k \dfrac{r}{R}\right)}{\mu_k^2 J_1(\mu_k)} \cos \frac{a\mu_k t}{R}\right].$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive zeros of $J_0(x)$.

88. A homogeneous circular membrane of radius R fixed around the edge is in equilibrium while a tension T is present. At time $t = 0$ a uniformly distributed pressure $f = P_0 \sin \omega t$ is applied to one side of the membrane. Find the radial vibrations of the membrane.

89. A square homogeneous membrane fixed around the edge started vibrating without initial velocities and had the initial form $Axy(b - x)(b - y)$. Find the free vibrations of this membrane.

90. Find the free vibrations of a homogeneous membrane fixed around the edge, which are produced by a blow near the center of the membrane, such that

$$\lim_{\varepsilon \to 0} \iint_{\sigma_\varepsilon} v_0 \, dx \, dy = A$$

is, where v_0 is the initial velocity and A a constant.

91. Find a solution for the wave equation

$$\Delta u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$$

in the domain

$$t \geqslant 0,\ 0 < a \leqslant r \leqslant b,\ 0 \leqslant \varphi \leqslant 2\pi,\ 0 \leqslant \theta \leqslant \alpha < \pi;$$

which satisfies the boundary conditions

$$u|_{r=a} = u|_{r=b} = u|_{\theta=\alpha} = 0$$

and the initial conditions

$$u|_{t=0} = \varphi(r, \theta)\, e^{im\varphi},\quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \varphi_1(r, \theta)\, e^{im\varphi}$$

where m is a positive integer.

92. Investigate by means of the method of separation of variables the transversal vibrations of a doubly sustained bar of lenght l.

93. An e.m.f. E is applied to an electric distortion free conductor $(R/2 = G/c)$ of lenght l. The end $x = l$ is insulated and at time $t = 0$ the end $x = 0$ is earthed. Show that the potential in a point x is given by:

$$V = \frac{4E}{\pi} e^{-bt} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin\frac{2n+1}{2l}\pi x \cdot \cos\frac{2n+1}{2l}\pi a t$$

where

$$b = \frac{R}{L};\quad a^2 = \frac{1}{LC}.$$

94. The end $x = l$ of a conductor of lenght l and negligibly small loss $(G = 0)$ is insulated. The initial current and voltage equal zero. An e.m.f. E is applied at the end $x = 0$ at time $t = 0$.
Find the current and voltage at time $t > 0$.

2. Equations of parabolic type

95. Find the solution of the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l,\ t > 0,$$

which satisfies the conditions

$$u(0, t) = u(l, t) = 0, \quad t > 0,$$

$$u(x, 0) = \begin{cases} x & \text{for } 0 < x \leqslant \dfrac{l}{2}, \\[2ex] l - x & \text{for } \dfrac{l}{2} \leqslant x < l. \end{cases}$$

96. Find the temperature distribution at time $t > 0$ of a thin homogeneous rod of lenght l, thermally insulated, if its initial temperature distribution is given by

$$f(x) = \frac{cx(l - x)}{l^2}$$

and if its ends are maintained at temperature zero.

97. The temperature distribution of a sphere of radius R and with center at the origin is a function only of the distance r from the origin. At the surface a temperature zero is measured at all times of interest. Find the distribution of the temperature inside the sphere for $t > 0$.

98. Given a thin homogeneous rod of lenght l, whose sides are thermally isolated. The initial temperature is known. At the end $x = 0$ a temperature zero is maintained, while at the other end $x = l$ an exchange of heat takes places with the surroundings at temperature zero. Find the temperature distribution at time $t > 0$.

99. Solve problem 98 assuming that an exchange of heat takes place at both ends of the rod.

100. Solve problem 97 if it is assumed that the sphere is cooled in a medium at temperature zero.

101. A homogeneous sphere of radius R has a constant temperature u_0 at time $t = 0$ and is surrounded by a spherical shell of thickness R, made of the same material and whose temperature vanishes. Find the radial distribution of temperature in the sphere.

102. The initial temperature distribution of a rod, thermally isolated along the surface, is given by $u(x, 0) = \varphi(x)$. Find the temperature of the rod if also one of its ends is thermally insulated and if the temperature of the other end is maintained equal to u_0.

103. The initial temperature of a thin homogeneous rod of lenght l is equal to zero. The end $x = 0$ is maintained at zero temperature, while the temperature at the other end $x = 0$ increases linearly, such that $u(0, t) = At$, where A is a positive constant. Find the distribution of the temperature of the rod.

104. Solve problem **103** assuming that the temperature at the end $x = 0$ changes according to the law $v(0, t) = A \sin \omega t$.

105. Find a solution for the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (0 < x < l,\ t > 0),$$

which satisfies the initial condition

$$u(x, 0) = 0$$

and the boundary conditions

$$u|_{x=0} = A(1 - e^{-\alpha t}), \quad \frac{\partial u}{\partial x} + Hu|_{x=l} = 0$$

where A, $H > 0$ and $\alpha > 0$ are constant.

106. A homogeneous sphere of radius R has at time $t = 0$ zero temperature and is surrounded by a medium of the same temperature. At time $t = 0$ the temperature of this medium starts to increase linearly according to $u = bt$, where $b > 0$ is constant. The exchange of heat takes place obeying Newton's law. The heating is uniform (spherical symmetry). Find the distribution of temperature at time $t > 0$ inside the sphere as a function of r, the distance to the origin.

107. An infinite plane layer of thickness $2R$ with initial temperature zero is uniformly heated on both sides by a constant flow of heat q. Compute for an arbitrary time $t > 0$ the temperature distribution as a function of the thickness.

108. The initial temperature distribution of a thin homogeneous rod of lenght l is equal to $f(x)$. At the end $x = 0$ a constant temperature u_0 is maintained and at the other end $x = l$ a constant temperature u_1). At the surface of the rod heat is transfered by radiation to the environment, which is at zero temperature. Find the distribution of the temperature in the rod for time $t > 0$.

109. A thin homogeneous rod of lenght l has an initial temperature zero. The end $x = 0$ is maintained at a constant temperature u_0. At the end $x = l$ as well as at the surface heat is transfered by radiation to a medium with temperature zero. Find the temperature of the rod at time $t > 0$.

110. When the equation for the conduction of heat in a homogeneous ring with a very small cross-section is constructed, one can assume that an exchange of heat with the environment takes place at the surface.
Solve the equation if the initial temperature is known.

111. Find the temperature distribution of a conductor of lenght l, through which a constant electric current flows, if at the ends of the conductor the temperature is maintained equal to u_0 and u_1 respectively. At the surface an exchange of heat takes place with the environment at zero temperature. The initial temperature of the conductor is 0°.

112. Investigate the radial conduction of heat in an infinitely long circular cylinder of radius R, if the surface is maintained at a constant temperature u_0. The initial temperature of the cylinder is zero.

113. Find the temperature distribution of an infinitely long circular cylinder of radius R if the initial temperature is a function of r only: $f(r)$. At the surface of the cylinder heat is radiated to the surrounding medium which has the temperature 0°.

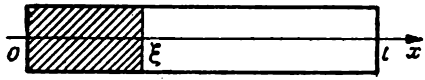

Fig. 6

114. A thin rod of lenght l is obtained by joining together two rods, made of different materials (fig. 6). At the end $x = 0$ of the rod the temperature is maintained constant and equal to u_0 while the other end is at zero temperature. The initial temperature of both rods equals zero. Find the temperature distribution at time $t > 0$.

115. The initial temperature of a thin rectangular plate with sides l and m is known. It is observed that at all times $t > 0$ the sides $x = 0$ and $x = l$ are at zero temperature, while the other sides have a given distribution of the temperature

$$u|_{y=0} = \varphi_0(x), \quad u|_{y=m} = \varphi_1(x), \quad 0 \leqslant x \leqslant l.$$

Determine the temperature of an arbitrary point of the plate at time $t > 0$.

116. The temperature distribution of a homogeneous circular cylinder of radius R and lenght l is $f(x, r)$. At time $t = 0$ the environment is at zero temperature. The exchange of heat at the surface with the surrounding medium obeys Newton's law. Find the distribution of the temperature inside the cylinder for arbitrary time $t > 0$.

117. Determine the solution of the equation

$$\frac{\partial^2 u}{\partial x^2} = \tfrac{3}{2}(1 - x^2)\,\frac{\partial u}{\partial t},$$

which satisfies

$$u(0, t) = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0, \quad u(x, 0) = 1.$$

118. Show that the solution of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

satisfying the boundary conditions

$$u(0, t) = P = \text{const}, \quad u(l, t) = 0 \quad (t > 0)$$

and the initial conditions

$$u(x, 0) = 0 \quad (0 < x < l)$$

can be expressed in the closed form:

$$u(x, t) = \frac{P(l-x)}{2l} +$$

$$+ \frac{P}{2\pi i} \int_0^\infty \frac{1}{\tau} \left[\frac{\sin (x-l)\sqrt{i\tau}}{\sin l\sqrt{i\tau}} e^{-t\tau i} - \right.$$

$$\left. - \frac{\sin (x-l)\sqrt{-i\tau}}{\sin l\sqrt{-i\tau}} e^{t\tau i} \right] d\tau$$

119. Find the solution of the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = a^2 \frac{\partial^2 u}{\partial x^2},$$

which satisfies the boundary conditions

$$u(0, t) = 0,$$
$$u(l, t) = 0 \quad (t > 0)$$

and the initial conditions

$$u|_{t=0} = f(x) \quad (0 \leqslant x \leqslant l).$$

120. Find the temperature of an semi-infinite rod at times $t > 0$, if the initial temperature is given and if at the end $x = 0$ the temperature is maintained equal to zero.

121. The initial temperature of a semi-infinite rod, thermally insulated along the surface, is known. At the end of the rod an exchange of heat takes place with the environment at zero temperature. Find the temperature distribution of the rod as a function of the length of the rod for arbitrary times $t > 0$.

122. An infinite rod is obtained by joining together a semi-infinite

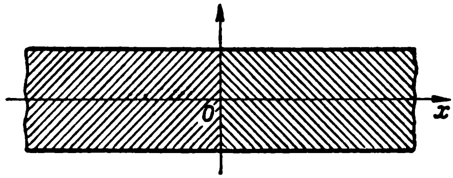

Fig. 7

rod at zero temperature and a semi infinite rod at temperature u_0. Find the temperature distribution in the two rods at time $t > 0$.

3. Equations of elliptic type

123. Find the solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the rectangle D: $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, which satisfies at the boundaries

$$u|_{x=0} = \varphi_0(y), \quad u|_{x=a} = \varphi_1(y), \quad 0 \leqslant y \leqslant b,$$
$$u|_{y=0} = \psi_0(x), \quad u|_{y=b} = \psi_1(x), \quad 0 \leqslant x \leqslant a$$

where

$$\varphi_0(0) = \psi_0(0), \quad \varphi_0(b) = \psi_1(0), \quad \varphi_0(0) = \psi_1(a), \quad \varphi_1(b) = \psi_1(a).$$

Solve this problem in the special case:

$$\varphi_0(y) = Ay(b - y), \quad \psi_0(x) = B \sin \frac{\pi x}{a}, \quad \varphi_1(y) = \psi_1(x) = 0.$$

124. Find the solution of Laplace's equation in the semi-infinite strip $0 \leqslant x \leqslant a$, $0 \leqslant y < \infty$, which satisfies the boundary conditions

$$u(0, y) = 0, \quad u(a, y) = 0, \quad u(x, 0) = A\left(1 - \frac{x}{a}\right),$$
$$u(x, \infty) = 0 \quad (0 \leqslant x \leqslant a).$$

125. Find the function, harmonic inside the ring $1 \leqslant r \leqslant 2$ and

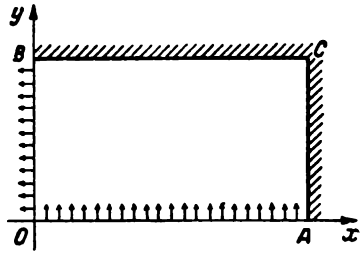

Fig. 8

satisfying the boundary conditions

$$u|_{r=1} = 0, \quad u|_{r=2} = Ay.$$

126. Solve the Dirichlet problem for Laplace's equation in the ring $R_1 \leqslant r \leqslant R_2$.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Investigate the case where R_1 tends to zero.

127. Find a solution for Laplace's equation in the ring $R_1 \leqslant r \leqslant R_2$, satisfying the boundary conditions

$$\left.\frac{\partial u}{\partial r}\right|_{r=R_1} = f_1(\theta), \quad u|_{r=R_2} = f_2(\theta).$$

128. Find the solution of Laplace's equation in the rectangle D: $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, satisfying the boundary conditions

$$u(0, y) = A, \quad u(a, y) = Ay,$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=b} = 0.$$

129. Find the function, harmonic inside the circular section $0 \leqslant \rho \leqslant R$, $0 \leqslant \theta \leqslant \alpha$, which satisfies

$$u(\rho, 0) = u(\rho, \alpha) = 0, \quad u(R, \varphi) = A\varphi.$$

130. Determine the function, harmonic inside a sphere of radius 1 and which assumes the values $\varphi(\theta) = \cos^2\theta$ on the sphere.

131. The sides AC and BC of a rectangular plate $OACB$ (fig. 8)

are thermally insulated. Through the side OA a uniform constant flow of heat into the rectangle takes place and through the side OB a uniform constant flow of heat into the environment. Find then the stationary temperature distribution of the plate.

132. Find a solution u of Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2$$

in the rectangle D: $0 \leqslant x \leqslant a$, $-b/2 \leqslant y \leqslant +b/2$, vanishing on the boundary.

133. Find a solution of Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -4$$

in a circle of radius a with center at the origin, if the solution satisfies:

$$u|_{r=a} = 0.$$

134. Find a solution of Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -xy$$

in a circle of radius a, and vanishing on the boundary.

$$u|_{r=a} = 0.$$

135. Solve the boundary-value problem for Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 12(x^2 - y^2)$$

in the ring $a \leqslant r \leqslant b$ and with boundary conditions

$$u|_{r=a} = 0, \quad \left.\frac{\partial u}{\partial r}\right|_{r=b} = 0.$$

136. The lateral sides and the bottom of a straight circular cylinder of height l and radius R are at zero temperature. The temperature

distribution of the upper side is a function of r alone. Find the stationary temperature distribution inside the cylinder.

137. The lateral sides of a straight circular cylinder of height h and radius R are thermally insulated. The temperature of the bottom equals zero and the distribution of the temperature of the upper side is a function of r alone. Find the stationary temperature distribution inside the cylinder.

138. Solve problem 136, assuming that the bottom of the cylinder is cooled in air at zero temperature.

139. The lower and the upper side of a straight circular cylinder of radius R and height h are at temperature zero. The temperature of the lateral side is a known function of z. Find the stationary temperature distribution inside the cylinder.

140. Solve problem 139, assuming that the bottom of the cylinder is thermally insulated.

141. The bottom of a straight circular cylinder of radius R and height H is at temperature u_0. The lateral and upper sides of the cylinder are cooled freely in air at temperature zero. Find the stationary temperature distribution inside the cylinder.

142. Find the temperature distribution in a semi-shpere if the temperature of the bottom is equal to zero and the surface of the semi-sphere is at temperature u_0.

143. Solve the boundary-value problem for the equation

$$\Delta v + k^2 v = 0$$

in a sphere of radius R, if the boundary-conditions are

$$v|_{r=R} = f(\theta, \varphi).$$

144. Find the solution of the equation

$$\Delta v + k^2 v = 0$$

in the domain D: $0 < a \leqslant r \leqslant b$, $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant \theta \leqslant \alpha < \pi$, which satisfies the boundary condition

$$v|_{r=a} = v|_{r=b} = 0, \quad v|_{\theta=\alpha} = f(r)\, e^{im\varphi},$$

where m is a positive integer.

Part II

SOLUTIONS AND HINTS

1. $\dfrac{\partial^2 u}{\partial\xi\partial\eta} + \frac{1}{2}\dfrac{\partial u}{\partial\xi} = 0; \quad \xi = x + y, \quad \eta = 3x - y.$

2. $\dfrac{\partial^2 u}{\partial\xi^2} + \dfrac{\partial^2 u}{\partial\eta^2} + \dfrac{\partial u}{\partial\eta} = 0; \quad \xi = 2x - y, \quad \eta = x.$

3. $\dfrac{\partial^2 u}{\partial\eta^2} + (\alpha + \beta)\dfrac{\partial u}{\partial\xi} + \beta\dfrac{\partial u}{\partial\eta} + cu = 0, \quad \xi = x + y, \quad \eta = y.$

4. $\dfrac{\partial^2 u}{\partial\xi\partial\eta} + \dfrac{\eta - \xi}{32}\left(\dfrac{\partial u}{\partial\xi} - \dfrac{\partial u}{\partial\eta}\right) = 0;$

   $\xi = 2x + \sin x + y, \quad \eta = 2x - \sin x - y.$

5. $\dfrac{\partial^2 u}{\partial\xi^2} + \dfrac{\partial^2 u}{\partial\eta^2} + \dfrac{1}{\xi - \eta}\cdot\dfrac{\partial u}{\partial\xi} + \dfrac{1}{2\eta}\cdot\dfrac{\partial u}{\partial\eta} = 0;$

   $\xi = x^2 - y^2, \quad \eta = x^2.$

6. $\dfrac{\partial^2 u}{\partial\eta^2} - \dfrac{2\xi}{\eta^2}\cdot\dfrac{\partial u}{\partial\xi} = 0; \quad \xi = y\sin x, \quad \eta = y.$

7. $\dfrac{\partial^2 u}{\partial\xi^2} + \dfrac{\partial^2 u}{\partial\eta^2} + \dfrac{1}{3\eta}\cdot\dfrac{\partial u}{\partial y} = 0; \quad \xi = x, \eta = \frac{2}{3}y^{\frac{3}{2}} \quad (y > 0);$

   $\dfrac{\partial^2 u}{\partial\xi\partial\eta} - \dfrac{1}{6(\xi - \eta)}\left(\dfrac{\partial u}{\partial\xi} - \dfrac{\partial u}{\partial\eta}\right) = 0; \quad \xi = x - \frac{2}{3}(-y)^{\frac{3}{2}},$

   $\eta = x + \frac{2}{3}(-y)^{\frac{3}{2}} \quad (y < 0).$

8. $\dfrac{\partial^2 u}{\partial\xi\partial\eta} + \dfrac{1}{4\eta}\cdot\dfrac{\partial u}{\partial\xi} - \dfrac{1}{\xi}\cdot\dfrac{\partial u}{\partial\eta} + u = 0; \quad \xi = xy, \quad \eta = \dfrac{x^3}{y}.$

9. $\dfrac{\partial^2 u}{\partial\xi^2} + \dfrac{\partial^2 u}{\partial\eta^2} = 0;$

   $\xi = \ln(x + \sqrt{1 + x^2}; \quad \eta = \ln(y + \sqrt{1 + y^2}).$

10. $\dfrac{\partial^2 u}{\partial \eta^2} - \dfrac{2\xi}{\xi^2 + \eta^2} \cdot \dfrac{\partial u}{\partial \xi} = 0; \quad \xi = y \operatorname{tg} \dfrac{x}{2}, \quad \eta = y.$

11. $\dfrac{\partial^2 u}{\partial \xi \partial \eta} - \dfrac{\alpha - \frac{1}{2}}{\xi - \eta}\left(\dfrac{\partial u}{\partial \xi} - \dfrac{\partial u}{\partial \eta}\right) = 0, \quad \xi = x - 2\sqrt{-y},$

$$\eta = x + 2\sqrt{-y} \quad (y < 0);$$

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{2\alpha - 1}{\eta} \cdot \frac{\partial u}{\partial \eta} = 0, \quad \xi = x, \ \eta = 2\sqrt{y} \ (y > 0).$$

12. $u(x, y) = \varphi(x + y - \cos x) + \psi(x - y + \cos x).$

13. $u(x, y) = \varphi(xy) \ln y + \psi(xy).$

14. $u(x, y) = \dfrac{\varphi(x - y) + \psi(x + y)}{x}.$

Hint: Introduce a new function v, defined by $v = x \cdot u$.

15. $u(x, y) = \dfrac{X(x) - Y(y)}{x - y}$, where $X(x)$ and $Y(y)$ are arbitrary functions.

Hint: Introduce a new function v:

$$v = (x - y)u.$$

16. $u(x, y, z) = (z - y)\varphi\left(\dfrac{y}{x}, \dfrac{z}{x}\right) + \psi\left(\dfrac{y}{x}, \dfrac{z}{x}\right)$; where φ and ψ are arbitrary functions.

Hint: Introduce new independent variables ξ, η and ξ defined by

$$\xi = \frac{y}{x}, \quad \eta = \frac{z}{x}, \quad \zeta = z - y.$$

17. $u(x, y, t) = \varphi(x + \sqrt{a_{11}}t, \quad y + \sqrt{a_{22}}t) +$

$$+ \psi(x - \sqrt{a_{11}}t, \quad y - \sqrt{a_{22}}t),$$

where φ and ψ are arbitrary functions.

18. $u(x, y) = (x - y)f_1(x + y) + (x + y)f_2(x - y) +$

$$+ f_3(x - y) + f_4(x + y),$$

where f_1, f_2, f_3 and f_4 are arbitrary functions.

19. $b^2 - ac = 1 - x^2 + y^2$.

For $1 - x^2 + y^2 > 0$ the equation is of hyperbolic type, for $1 - x^2 + y^2 < 0$ of elliptic type. On the curve $x^2 - y^2 = 1$ the equation is of parabolic type.

$$u(x, y) = \varphi\left(\frac{y + \sqrt{1 - x^2 + y^2}}{1 + x}\right) + \psi\left(\frac{y - \sqrt{1 - x^2 + y^2}}{1 + x}\right).$$

Hint: Introduce new variables (z, t) by $t^2 = 1 - x^2$, $y = zt$ to integrate the equations of the characteristic curves

$$(xy \pm \sqrt{1 - x^2 + y^2})\,dx + (1 - x^2)\,dy = 0.$$

20. If $b_1 = b_2 = 0$, the equation $L(u) = 0$ has two functional-invariant solutions and its general solution is

$$u(x, y) = \psi_1(\alpha_1 x - y) + \psi_2(\alpha_2 x - y),$$

where ψ_1 and ψ_2 are arbitrary functions and a_1 and a_2 solutions of the equation

$$a_{11}\alpha^2 - 2a_{12}\alpha + a_{22} = 0.$$

If $b_1 \neq 0$, $b_2 \neq 0$ and $\alpha_1 = b_2/b_1$ the equation $L(u) = 0$ has only one functional invariant solution the general form of which is given by

$$u(x, y) = e^{[a_{11}(\alpha_2 b_1 - b_2)(\alpha_1 x - y)]/4\delta}\,\varphi_1(\alpha_2 x - y) + \varphi_2(\alpha_1 x - y),$$

where φ_1 and φ_2 are arbitrary functions. For the case that the equation $L(u) = 0$ has non-constant coefficients see the paper of N. P. Erugin.[1]

21. Hint: Transform the equation $L(u) + Cu = 0$ to the canonical form and try a solution of the form $u = v \cdot w$. See also the paper of N. P. Erugin.[1]

22. Hint: Apply the method of successive approximations. See also the paper of N. P. Erugin.[1]

[1] (N. P. Erugin, Functional-invariant solutions of a partial differential equation of second order in two independent variables. Scientific Papers of the Leningrad State University. Mathematical Series Volume 16, 1949).

23. Hint: Differentiate the equation

$$(x-y)\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

$(m - 1)$ times with respect to x and $(n - 1)$ times with respect to y.

24. $$u(x, y) = \frac{1}{x - y} \frac{\partial}{\partial x}\left[\frac{X(x) - Y(y)}{x - y}\right];$$

where $X(x)$ and $Y(y)$ are arbitrary functions.
Hint: Introduce a function v by the substitution

$$u = v(x - y)^{-1}.$$

25. The equation $E(\alpha, \beta) = 0$ has special solutions of the form $(a - x)^{-\alpha}(y - a)^{-\beta}$, where a is constant. Also

$$u(x, y) = \int_x^y \varphi(z)(z - x)^{-\alpha}(y - z)^{-\beta}\,dz,$$

where $\varphi(z)$ is an arbitrary function, is a solution of the equation $E(\alpha, \beta) = 0$.
If $Z(\alpha, \beta)$ is a solution of the equation $E(\alpha, \beta) = 0$ then $Z(\alpha, \beta)$ satisfies

$$(y - x)^{1-\alpha-\beta}Z(1 - \beta, 1 - \alpha). \qquad (*)$$

The function $\int_x^y \psi(z)(z - x)^{\beta-1}(y - z)^{\alpha-1}\,dz$, where $\psi(z)$ is an arbitrary function, is a solution of the equation $E(1 - \beta, 1 - \alpha) = 0$, and by (*) we get for the equation $E(\alpha, \beta) = 0$

$$(y - x)^{1-\alpha-\beta}\int_x^y \psi(z)(z - x)^{\beta-1}(y - z)^{\alpha-1}\,dz.$$

The general solution of the equation $E(\alpha, \beta) = 0$ is given by

$$u(x, y) = \int_x^y \psi(z)(z - x)^{-\alpha}(y - z)^{-\beta}\,dz +$$

$$+ (y - x)^{1-\alpha-\beta}\int_x^y \psi(z)(z - x)^{\beta-1}(y - z)^{\alpha-1}\,dz.$$

After the substitution $z = x(1 - t) + yt$, we find the required solution of the equation $E(\alpha, \beta) = 0$.

26. Hint: Substitution of $\alpha = \alpha' + m$, $\beta = \beta' + m$ in the formula (*) of problem 25 and investigation of the identity

$$Z(\alpha' + m,\ \beta' + n) = \frac{\partial^{m+n}}{\partial x^m \partial y^n} Z(\alpha',\ \beta'),$$

gives

$$(x - y)^{1-m-n-\alpha'-\beta'} Z(1 - \beta' - n,\ 1 - \alpha' - m) = \frac{\partial^{m+n} Z(\alpha',\ \beta')}{\partial x^m \partial y^n}.$$

If the formula (*) is used again, the result will be

$$(x - y)^{1-m-n-\alpha'-\beta'} Z(1 - \beta' - n,\ 1 - \alpha' - m) =$$
$$= \frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[\frac{Z(1 - \beta',\ 1 - \alpha')}{(x - y)^{\alpha'+\beta'-1}} \right].$$

If $1 - \beta'$, $1 - \alpha'$, n and m are substituted for α', β', m and n respectively, this becomes $Z(\alpha' - m,\ \beta' - n) =$

$$= (x - y)^{m+n+1-\alpha'-\beta'} \frac{\partial^{m+n}}{\partial x^n \partial y^m} \left[\frac{Z(\alpha',\ \beta')}{(x - y)^{1-\alpha'-\beta'}} \right].$$

For $\alpha' = \beta' = 0$ we get

$$Z(-m,\ -n) = (x - y)^{m+n+1} \frac{\partial^{m+n}}{\partial x^n \partial y^m} \left[\frac{X(x) - Y(y)}{x - y} \right].$$

27. $$u(x, y) = 8^{1-\alpha}(-y)^{1-\alpha} \int_0^1 \Phi[x - 2\sqrt{-y}(1 - 2t)] t^{\frac{1}{2}-\alpha} \times$$
$$\times (1 - t)^{\frac{1}{2}-\alpha} dt + \int_0^1 \psi[x - 2\sqrt{-y}(1 - 2t)] t^{\alpha-\frac{3}{2}} (1 - t)^{\alpha-\frac{3}{2}} dt$$
$(y < 0)$

where Φ and ψ are arbitrary functions.
Hint: See problems **11** and **25.**

28. a) $u(x, t) = \sqrt{x} \cdot F_1(\ln x - t)$;

b) $u(x, t) = e^{-x} F_2(x^2 - t)$,

where F_1 and F_2 are arbitrary functions.
Hint: Assume that the solution has the form

$$u(x, t) = \Phi(x) F[\omega(x) - t].$$

29. a) $u(x\, y, t) = \dfrac{\mathfrak{Cof}\sqrt{c\rho}}{\rho}$,

with

$$\rho = \sqrt{(t-\tau)^2 - (x-\xi)^2 - (y-\eta)^2};$$

b) $u(x, y, z, t) = \dfrac{J_0(\sqrt{-c\rho})}{\rho^2} + \dfrac{\sqrt{-c}J'_0(\sqrt{-c\rho})}{\rho} \ln \rho$,

with

$$\rho = \sqrt{(t-\tau)^2 - (x-\xi)^2 - (y-\eta)^2 - (z-\zeta)^2}.$$

Hint: Find a solution only depending on ρ.

30. $u(x, y) = 3x^2 + y^2$.

Hint: The solution is a special case of the general solution

$$u(x, y) = \varphi(x+y) + \psi(3x-y)$$

of the given equation.

31. $$u(x, y) = \frac{\varphi_0\left(\dfrac{\alpha^2-1}{2\alpha}\right) + \varphi_0\left(\dfrac{\beta^2-1}{2\beta}\right)}{2} - $$

$$-\frac{1}{2}\int_\alpha^\beta \frac{1}{z}\varphi_1\left(\frac{z^2-1}{2z}\right)dz$$

with

$$\alpha = (x+\sqrt{1+x^2})(y+\sqrt{1+y^2}), \quad \beta = \frac{x+\sqrt{1+x^2}}{y+\sqrt{1+y^2}}$$

32. $$u(x, y) = \frac{\varphi_0(x - \sin x + y) + \varphi_0(x + \sin x - y)}{2} +$$

$$+ \tfrac{1}{2}\int_{x+\sin x-y}^{x-\sin x+y} \varphi_1(z)\, dz.$$

33. $$u(x, y) = \tfrac{3}{4}\varphi_0(x\sqrt[3]{y}) + \tfrac{1}{4}y\varphi_0\left(\frac{x}{y}\right) +$$

$$+ \tfrac{3}{16}\sqrt[4]{x^3y}\int_{x\sqrt[3]{y}}^{x/y} \varphi_0(x)x^{-7/4}dx - \tfrac{3}{4}\sqrt[4]{x^3y}\int_{x\sqrt[3]{y}}^{x/y} \varphi_1(x)x^{-7/4}dx.$$

34. $$u(r, t) = \frac{(r - at)\varphi(r - at) + (r + at)\varphi(r + at)}{2r} + $$

$$+ \frac{1}{2ar} \int\limits_{r-at}^{r+at} \rho\psi(\rho)\, d\rho.$$

Hint: Introduce spherical coördinates. Obviously the solution is only a function of the radius and time. Therefore the wave equation reduces to

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\,\frac{\partial u}{\partial r}\right).$$

The general solution of this equation is

$$u(r, t) = \frac{\theta_1(r - at) + \theta_2(r + at)}{r},$$

where θ_1 and θ_2 are arbitrary functions.

35. $u(x, y) =$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{y}{\Gamma(\frac{7}{8})\Gamma(\frac{5}{8})} \int\limits_0^1 \varphi_1[x + y^2(t - \tfrac{1}{2})]t^{-\frac{1}{8}}(1 - t)^{-\frac{3}{8}} dt +$$

$$+ \frac{\sqrt{\pi}}{\Gamma(\frac{3}{8})\Gamma(\frac{1}{8})} \int\limits_0^1 \varphi_0[x + y^2(t - \tfrac{1}{2})]t^{-\frac{5}{8}}(1 - t)^{-\frac{7}{8}} dt,$$

with

$$\Gamma(s) = \int\limits_0^\infty \mathrm{e}^{-x} x^{s-1}\, dx \quad \text{for } s > 0.$$

Hint: Reduce the equation to canonical form and use the general solution of problem 25 for

$\alpha = \frac{1}{8}$ and $\beta = \frac{3}{8}$.

36. $$u(x, y) = \frac{\tau(x - 2\sqrt{-y}) + \tau(x + 2\sqrt{-y})}{2} \quad (y < 0).$$

Hint: The solution is a special case of the general solution

$$u(x, y) = \theta_1(x - 2\sqrt{-y}) + \theta_2(x + 2\sqrt{-y})$$

of the given equation.

37. $u(x, y) =$

$$= -\frac{2\Gamma(2 - 2\alpha(-y)^{1-\alpha}}{\Gamma^2(\frac{3}{2} - \alpha)} \int_0^1 \nu[x - 2\sqrt{-y}(1 - 2t)]t^{\frac{1}{2}-\alpha} \times$$

$$\times (1 - t)^{\frac{1}{2}-\alpha} dt + \frac{\Gamma(2\alpha - 1)}{\Gamma^2(\alpha - \frac{1}{2})} \int_0^1 \tau[x - 2\sqrt{-y}(1 - 2t)]t^{\alpha-\frac{3}{2}} \times$$

$$\times (1 - t)^{\alpha-\frac{3}{2}} dt.$$

Hint: See problem **27.**

38. $v(x, y) = \int_0^x \psi_1(t)F(y(x - t))\,dt +$

$$+ \int_0^y \psi_2(t)F(x(y - t))\,dt + \varphi(0)F(xy),$$

with

$$\psi_1(x) - \psi_2(x) =$$

$$= \frac{d}{dx} \int_0^x [\omega_1(\tau) - \omega_2(\tau)]F(\tau(\tau - x))\,d\tau, \quad F(z) = \mathrm{J}_0(2i\sqrt{z}).$$

$$\psi_1(x) + \psi_2(x) =$$

$$= \frac{d}{dx}\left[\varphi(x) - x \int_0^x \varphi(\tau)F'(\tau(\tau - x))\,d\tau\right].$$

Hint: Use the general solution of problem **22.** See also the paper ot N. P. Eurugin.[1]

39. $u(x, y, t) = f(x - \sqrt{a_{11}}t, y - \sqrt{a_{22}}t) +$

$$+ \frac{1}{2\sqrt{a_{11}}} \int_{x-\sqrt{a_{11}}t}^{x+\sqrt{a_{11}}t} [\sqrt{a_{11}}f_x(x, y) + \sqrt{a_{22}}f_y(x, y) + F(x, y)]\,dx.$$

[1] See footnote for problem 20.

Put

$$y = \sqrt{\frac{a_{22}}{a_{11}}}x - \frac{c_1}{\sqrt{a_{11}}}.$$

Hint: See problem **17**.

$$u(x, y, t) = x^2 + y^2 + (a_{11} + a_{22})t^2.$$

40. $u(x, y) = 4[\tau(x + y) + \tau(x - y)] - 2y[\tau'(x + y) -$

$$- \tau'(x - y)] - 2y(\nu(x + y) + \nu(x - y)] + 6 \int_{x-y}^{x+y} \nu(t)\,dt +$$

$$+ 2y \int_{x-y}^{x+y} \nu_1(t)\,dt - \int_{x-y}^{x+y} [(x - t)^2 - y^2]\nu_2(t)\,dt.$$

Hint: See problem **18**.

41. $\tau = \dfrac{2al}{a^2 - v^2}$,

with $a = \sqrt{\dfrac{T}{\rho}}$ and where l is the distance between the pulleys.

42. $u(x, t) = \varphi\left(\dfrac{x - t}{2}\right) + \psi\left(\dfrac{x + t}{2}\right) - \varphi(0).$

Hint: The solution can be derived from the general solution

$$u(x, t) = \theta_1(x - t) + \theta_2(x + t).$$

43. $u(x, y) = \varphi_1\left(\dfrac{x + 2\sqrt{-y}}{2}\right) + \varphi_2\left(\dfrac{x - 2\sqrt{-y} + 1}{2}\right) -$

$$- \varphi_1(\tfrac{1}{2}).$$

Hint: The solution is a special case of the general solution

$$u(x, y) = \theta_1(x - 2\sqrt{-y}) + \theta_2(x + 2\sqrt{-y}) \quad (y < 0).$$

44. $u(x, y) = \varphi_1(x - 2\sqrt{-y}) - \varphi_2\left(\dfrac{x}{2} - \sqrt{-y}\right) +$

$$+ \varphi_2\left(\frac{x}{2} + \sqrt{-y}\right).$$

Hint: See the hint of problem **43**.

45. Solution:

$$u(x,t) = \varphi\left(\frac{x+t}{2}\right) - \varphi\left[\frac{f_1(x-t) + f(f_1(x-t))}{2}\right] + $$
$$+ \psi(f_1(x-t)),$$

with $z = x - f(x)$: and therefore $x = f_1(z)$.
Special case:

$$u(x,t) = \varphi\left(\frac{x+t}{2}\right) - \varphi\left[\frac{(1+k)(x-t)}{2(1-k)}\right] + \psi\left(\frac{x-t}{1-k}\right).$$

Hint: Integrate the given equation subject to the conditions:

$u(x,t) = \varphi(x)$ on the characteristic curve $x - t = 0$;

$u(x,t) = \psi(x)$ on the curve L, with $t = f(x)$, with $\varphi(0) = \psi(0)$.

46. The solution

$$u_1(x,t) = \frac{\varphi_0(x-t) + \varphi_0(x+t)}{2} + \tfrac{1}{2}\int\limits_{x-t}^{x+t} \varphi_1(z)\,dz$$

satisfies the initial conditions. Its values on the characteristic curve $x - t = 0$ are

$$\varphi(x) = \frac{\varphi_0(0) + \varphi_0(2x)}{2} + \tfrac{1}{2}\int\limits_{0}^{2x} \varphi_1(z)\,dz.$$

The solution

$$u_2(x,t) = \psi\left(\frac{x-t}{1-k}\right) + \frac{\varphi_0(x+t) - \varphi_0\left(\dfrac{1+k}{1-k}(x-t)\right)}{2} +$$
$$+ \frac{1}{2}\int\limits_{(1+k/1-k)(x-t)}^{x+t} \varphi_1(z)\,dz$$

transforms on the straight line $t = kx$ into $\psi(x)$, on the characteristic curve $x - t = 0$ it equals $\varphi(x)$.
A necessary and sufficient condition that not only u_1 and u_2

themselves be equal, but also their first order derivatives on the characteristic curve $x - t = 0$, is

$$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n}$$

where n is the normal to the characteristic curve $x - t = 0$. This is the case when in addition the following condition is satisfied:

$$\varphi_0'(0) + k\varphi_1(0) = \psi'(0).$$

47. For values of t, in the interior of the segments $\left(0, \dfrac{r-R}{a}\right)$ and $\left(\dfrac{r+R}{a}, \infty\right)$, the density equals zero.
For $\dfrac{r-R}{a} < t < \dfrac{r+R}{a}$ the density equals $u = \dfrac{u_0(r-at)}{2r}$.
Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

subject to the initial conditions

$$u|_{t=0} = \begin{cases} u_0 & \text{for } r < R \\ 0 & \text{for } r > R, \end{cases} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

Use also the hint of problem **34.**

48. Hint: Apply the method of characteristics to integrate the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions

$$u|_{x=0} = A \cdot \sin \omega t$$

and the initial conditions

$$u(x; 0) = 0; \quad \frac{\partial u(x, 0)}{\partial t} = 0 \text{ for } x > 0.$$

49. Hint: Apply the method of characteristics to integrate the

equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions

$$u|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0 \quad (t > 0)$$

and the initial conditions

$$u|_{t=0} = \varphi_0(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \varphi_1(x) \quad (0 < x < l).$$

The functions $\varphi_0(x)$ and $\varphi_1(x)$ can be continued from the segment $(0, l)$ to the segment $(l, 2l)$ by means of the following formulas:

$$\varphi_0(x + l) = \varphi_0(l - x), \quad \varphi_1(x + l) = \varphi_1(l - x).$$

Now the functions y_1 and y_2 can be defined on the whole interval $(-2l, 2l)$ as anti-symmetric functions and therefore with period $4l$.

51. $$u(x, t) = \begin{cases} \dfrac{aMv_0}{\gamma p_0 S}\left[1 - e^{(\gamma p_0 S/Ma^2)(x-at)}\right] & \text{for } x - at < 0 \\ 0 & \text{for } x - at > 0, \end{cases}$$

where p_0 is the initial pressure, S a cross-sectional area of the tube and $\gamma = c_p/c_v$.
Hint: This problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (x > 0)$$

subject to the conditions

$$M \frac{\partial^2 u(0, t)}{\partial t^2} = S\gamma p_0 \frac{\partial u(0, t)}{\partial x};$$

$$u(x, 0) = 0 \ (x \geqslant 0); \quad \frac{\partial u(x, 0)}{\partial t} = 0 \ (x > 0); \quad \frac{\partial u(0, 0)}{\partial t} = v_0.$$

52. Because of the symmetry one can limit oneself to the investigation of the string only for $x \geqslant 0$ and one can apply the method

of characteristics to the integration of the equation

$$\frac{\partial^2 u_1}{\partial t^2} = a^2 \frac{\partial^2 u_1}{\partial x^2} \quad (x \geqslant 0)$$

subject to the boundary conditions

$$M \left. \frac{\partial^2 u_1}{\partial t^2} \right|_{x=0} = 2T_0 \left. \frac{\partial u_1}{\partial x} \right|_{x=0}$$

and the initial conditions

$$u_1(x, 0) = 0 \ (x \geqslant 0); \quad \frac{\partial u_1(x, 0)}{\partial t} = 0 \ (x > 0); \quad \frac{\partial u_1(0, 0)}{\partial t} = v_0.$$

53. For the first cylinder $\frac{\partial u_1(x, t)}{\partial t}$ is a periodic function with period $T = \frac{4l}{a}$:

$$\frac{\partial u_1(x, t)}{\partial t} = \begin{cases} v_1 & \text{for} \quad 0 < t < \frac{l-x}{a}, \\ \frac{1}{2}(v_1 + v_2) & ,, \quad \frac{l-x}{a} < t < \frac{l+x}{a}, \\ v_2 & ,, \quad \frac{l+x}{a} < t < \frac{3l-x}{a}, \\ \frac{1}{2}(v_1 + v_2) & ,, \quad \frac{3l-x}{a} < t < \frac{3l+x}{a} \end{cases}$$

etc.

For the second cylinder we get

$$\frac{\partial u_2(x, t)}{\partial t} = \begin{cases} v_2 & \text{for} \quad 0 < t < \frac{l-x}{a}, \\ \frac{1}{2}(v_1 + v_2) & ,, \quad \frac{l-x}{a} < t < \frac{3l-x}{a}, \\ v_1 & ,, \quad \frac{3l-x}{a} < t < \frac{l+x}{a}, \\ \frac{1}{2}(v_1 + v_2) & ,, \quad \frac{l+x}{a} < t < \frac{5l-x}{a} \end{cases}$$

etc. where $u_1(x, t)$ is the longitudinal displacement of a cross-section of the first cylinder and $u_2(x, t)$ that of the second cylinder. Hint: Integrate the equation:

$$\frac{\partial^2 u_1}{\partial t^2} = a^2 \frac{\partial^2 u_1}{\partial x^2}, \quad \frac{\partial^2 u_2}{\partial t^2} = a^2 \frac{\partial^2 u_2}{\partial x^2}, \quad \text{where } a = \sqrt{\frac{E}{\rho}}$$

(E is the modulus of elasticity, $\rho =$ the density of the cylinder) subject to the boundary conditions

$$\frac{\partial u_1(0, t)}{\partial x} = 0; \quad \frac{\partial u_2(2l, t)}{\partial x} = 0;$$

$$u_1(l, t) = u_2(l, t), \quad \frac{\partial u_1(l, t)}{\partial x} = \frac{\partial u_2(l, t)}{\partial x}$$

and the initial conditions

$$u_1(x, 0) = 0, \quad u_2(x, 0) = 0;$$

$$\frac{\partial u_1(x, 0)}{\partial t} = v_1, \quad \frac{\partial u_2(x, 0)}{\partial t} = v_2.$$

54. $$\frac{\partial u}{\partial t} = a[f'(at - x) - f'(at + x)]$$

with

$$f'(z) = \frac{v}{a} e^{-k(z-l)} + \frac{v}{a} [1 - 2k(z - 3l)] e^{-k(z-3l)}, \quad k = \frac{E\omega}{Ma^2}.$$

E is the modulus of elasticity, M the mass of the falling load, ω the crosssectional area of the rod and ρ its density.
Hint: Integrate the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions

$$u(0, t) = 0, \quad M \frac{\partial^2 u(l, t)}{\partial t^2} = - E\omega \frac{\partial u(l, t)}{\partial x};$$

$$u(x, 0) = 0 \quad (0 \leqslant x \leqslant l),$$

$$\frac{\partial u(x, 0)}{\partial t} = 0 \quad (0 \leqslant x < l), \quad \frac{\partial u(0, l)}{\partial t} = - v.$$

55. See the paper of N. P. Erugin.[1]

56. $$T(x, y) = e^{-ay-bx}\Big[b^2\int_0^x e^{bt}\ \mathrm{J}_0(2i\sqrt{aby(x-t)})\,dt + \mathrm{J}_0(2i\sqrt{abxy})\Big],$$

$$\theta(x, y) = -ie^{-ay-bx}\Big[b^2\int_0^x e^{bt}\ \mathrm{J}_1(2i\sqrt{aby(x-t)}) \times \frac{x-t}{\sqrt{aby(x-t)}}\,dt + \frac{bx\mathrm{J}_0(2t\sqrt{abxy})}{\sqrt{abxy}}\Big].$$

Hint: After elimination of $\theta(x, y)$ we find for $T(x, y)$ the equation

$$\frac{\partial^2 T}{\partial x \partial y} + a\frac{\partial T}{\partial x} + b\frac{\partial T}{\partial y} = 0$$

with the conditions

$$T|_{x=0} = e^{-ay}, \quad T|_{y=0} = 1.$$

Putting

$$T(x, y) = e^{-ay-bx}u, \quad \xi = ax, \quad \eta = by,$$

we get

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = u, \quad u|_{\xi=0} = 1, \quad u|_{\eta=0} = e^{(b/a)\xi}.$$

Use also the general solution of problem 22.

57. $$u(x, t) = \varphi\left(t - \frac{x}{a}\right) - \varphi\left(t + \frac{x}{a}\right),$$

$$v(x, t) = \frac{1}{\rho}\left[\varphi\left(t - \frac{x}{a}\right) + \varphi\left(t + \frac{x}{a}\right)\right],$$

with

$$\varphi(x) \equiv 0 \quad \text{in} \quad \left(-\frac{l}{a}, \frac{l}{a}\right),$$

$$\varphi\left(t + \frac{l}{a}\right) = \frac{\rho[\alpha(t) - 1]}{1 + \beta\rho\alpha(t)} + \frac{\beta\rho\alpha(t) - 1}{\beta\rho\alpha(t) + 1}\,\varphi\left(t - \frac{l}{a}\right).$$

[1] See the footnote for problem 20.

Hint: Reduce the system of equations of first order to a single equation of second order.

58. $$u(x, t) = \frac{1536}{5\pi^5} \sum_{n=0}^{\infty} \frac{\cos \dfrac{(2n+1)\pi at}{l} \sin \dfrac{(2n+1)\pi x}{l}}{(2n+1)^5}.$$

59. $$u(x, t) = \frac{32h}{\pi^3} \sum_{n=0}^{\infty} \frac{\cos \dfrac{(2n+1)\pi at}{l} \sin \dfrac{(2n+1)\pi x}{l}}{(2n+1)^3},$$

with $h = u\left(\dfrac{l}{2}, 0\right)$.

Hint: Apply Fourier's method to integrate the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions

$$u(0, t) = 0, \quad u(l, t) = 0;$$

$$u(x, 0) = \frac{4hx(l-x)}{l^2}, \quad \frac{\partial u(x, 0)}{\partial t} = 0 \quad (0 \leqslant x \leqslant l).$$

60. $$u(x, t) = \frac{2hl^2}{\pi^2 c(l-c)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi c}{l} \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}.$$

Hint: Replace the first initial condition in problem **59** by the condition:

$$u(x, 0) = \begin{cases} \dfrac{h}{c} x & \text{for } 0 \leqslant x \leqslant c \\[2ex] \dfrac{h(x-l)}{c-l} & \text{for } c \leqslant x \leqslant l. \end{cases}$$

61. a) $$u(x, t) = \frac{4v_0 l}{\pi^2 a} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi c}{l} \sin \frac{n\pi^2}{2hl} \sin \frac{n\pi at}{l} \sin \frac{n\pi x}{l};$$

b) $$u(x, t) = \frac{4hv_0}{\pi^2 a} \sum_{n=1}^{\infty} \frac{\sin \dfrac{n\pi c}{l} \cos \dfrac{n\pi^2}{2hl}}{n\left(h^2 - \dfrac{n^2\pi^2}{l^2}\right)} \sin \frac{n\pi at}{l} \sin \frac{n\pi x}{l}.$$

62. $u(x, t) = \dfrac{8h}{\pi^2} \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)^2} \cos \dfrac{(2n+1)\pi at}{2l} \sin \dfrac{(2n+1)\pi x}{2l}$.

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions

$$u(0, t) = 0, \quad \frac{\partial u(l, t)}{\partial x} = 0,$$

$$u(x, 0) = \frac{hx}{l}, \quad \frac{\partial u(x, 0)}{\partial t} = 0.$$

63. $u(x, t) = \dfrac{8vl}{a\pi^2} \sum\limits_{n=0}^{\infty} \dfrac{1}{(2n+1)^2} \sin \dfrac{(2n+1)\pi at}{2l} \sin \dfrac{(2n+1)\pi x}{2l}$.

Hint: Replace the initial conditions in problem **62** by the conditions

$$u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = v.$$

64. $u(x, t) = \sum\limits_{n=0}^{\infty} \left[a_n \cos \dfrac{(2n+1)\pi at}{2l} + \right.$

$$\left. + \, b_n \sin \frac{(2n+1)\pi at}{2l} \right] \sin \frac{(2n+1)\pi x}{2l},$$

with

$$a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{(2n+1)\pi x}{2l} \, dx,$$

$$b_n = \frac{4}{(2n+1)a\pi} \int_0^l F(x) \sin \frac{(2n+1)\pi x}{2l} \, dx.$$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad a^2 = \frac{E}{\rho}$$

subject to the conditions

$$u(0, t) = 0, \quad \frac{\partial u(l, t)}{\partial x} = 0 \quad (t > 0);$$

$$u(x, 0) = f(x), \quad \frac{\partial u(x, 0)}{\partial t} = F(x) \quad (0 \leqslant x \leqslant l).$$

u is the displacement of the cross-section with abscis x; l is the length of the rod, ρ the linear density and E the modulus of elasticity.

65. $$u(x, t) = \frac{8Ql}{E\sigma\pi^2} \sum_{n=0}^{\infty} (-1)^n \frac{\cos\frac{(2n+1)\pi at}{2l} \sin\frac{(2n+1)\pi x}{2l}}{(2n+1)^2}.$$

Hint: See problem **64.**

66. $$u(x, t) = \frac{1}{l} \int_0^l [\varphi_0(x) + t\varphi_1(x)]\, dx +$$

$$+ \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi at}{l} + b_n \sin\frac{n\pi at}{l}\right) \cos\frac{n\pi x}{l},$$

with

$$a_n = \frac{2}{l} \int_0^l \varphi_0(x) \cos\frac{n\pi x}{l}\, dx, \quad b_n = \frac{2}{n\pi a} \int_0^l \varphi_1(x) \cos\frac{n\pi x}{l}\, dx.$$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions

$$\frac{\partial u(0, t)}{\partial x} = 0, \quad \frac{\partial u(l, t)}{\partial x} = 0;$$

$$u(x, 0) = \varphi_0(x), \quad \frac{\partial u(x, 0)}{\partial t} = \varphi_1(x) \quad (0 < x < l).$$

67. Hint: Integrate the equation of problem **66** subject to the

conditions

$$\frac{\partial u(-l, t)}{\partial x} = 0, \quad \frac{\partial u(l, t)}{\partial x} = 0;$$

$$u(x, 0) = -\varepsilon x, \quad \frac{\partial u(x, 0)}{\partial t} = 0.$$

68. $$\theta(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{\mu_n at}{l} + b_n \sin \frac{\mu_n at}{l}\right) \sin \frac{\mu_n x}{l},$$

$$a_n = \frac{4}{2\mu_n + \sin 2\mu_n} \int_0^l \varphi_0'(x) \cos \frac{\mu_n x}{l}\, dx,$$

$$b_n = \frac{4l}{a\mu_n(2\mu_n + \sin 2\mu_n)} \int_0^l \varphi'(x) \cos \frac{\mu_n x}{l}\, dx,$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the transcendental equation

$$\mu \operatorname{tg} \mu = \gamma \quad \left(\gamma = I\frac{k}{k_1}\right).$$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2\theta}{\partial t^2} = a^2 \frac{\partial^2\theta}{\partial x^2}, \quad a = \sqrt{\frac{GI}{k}}$$

subject to the conditions

$$\theta(0, t) = 0, \quad \frac{\partial^2\theta(l, t)}{\partial t^2} = -c^2 \frac{\partial\theta(l, t)}{\partial x}, \quad c = \sqrt{\frac{GI}{k_1}};$$

$$\theta(x, 0) = \varphi_0(x), \quad \frac{\partial\theta(x, 0)}{\partial t} = \varphi_1(x), \quad 0 < x < l.$$

θ is the angle of deflection of a cross-section with abscis x; G is the shear modulus, I is the geometric moment of inertia of a cross-section of the cylinder about its axis, k is the mechanical moment of inertia per unit length of the rod and k_1 the moment of inertia of the disk about the axis of rotation.

69. $$u(x, t) = \frac{Ql}{E\sigma}\left[\frac{x}{l} - 2\alpha^2 \sum_{n=1}^{\infty} \frac{\left(1 - \cos\dfrac{\beta_n at}{l}\right)\sin\dfrac{\beta_n x}{l}}{\beta_n^2 \sin\beta_n(\alpha + \alpha^2 + \beta_n^2)}\right],$$

where $\beta_1, \beta_2, \beta_3, \ldots$ are the positive roots of

$$\beta \operatorname{tg} \beta = \alpha \quad \left(\alpha = \frac{\rho l}{m}\right).$$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions

$$u(0, t) = 0, \quad m\frac{\partial^2 u(l, t)}{\partial t^2} = -E\sigma\frac{\partial u(l, t)}{\partial x};$$

$$u(x, 0) = \frac{Qx}{E\sigma}, \quad \frac{\partial u(x, 0)}{\partial t} = 0$$

70. $$u(x, t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{\mu_n at}{l} + b_n \sin\frac{\mu_n at}{l}\right)X_n(x),$$

with

$$a_n = \frac{\int_0^l \varphi_0(x)X_n(x)\,dx}{\int_0^l X_n^2(x)\,dx}, \quad b_n = \frac{l}{\mu_n a}\cdot\frac{\int_0^l \varphi_1(x)X_n(x)\,dx}{\int_0^l X_n^2(x)\,dx},$$

$$X_n(x) = \cos\frac{\mu_n x}{l} + \frac{2h_1 l}{T_0\mu_n}\sin\frac{\mu_n x}{l},$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of

$$\operatorname{ctg}\mu = \alpha\left(\frac{\mu}{l} - \frac{4h_1h_2l}{T_0^2\mu}\right) \quad \left(\alpha = \frac{T_0}{2(h_1 + h_2)}\right).$$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions

$$\left.\frac{\partial u}{\partial x} - \frac{2h_1}{T_0} u\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x} + \frac{2h_2}{T_0} u\right|_{x=l} = 0;$$

$$u(x, 0) = \varphi_0(x), \quad \frac{\partial u(x, 0)}{\partial t} = \varphi_1(x)$$

where h_1 and h_2 are positive constants.

71. $$u(x, t) = e^{-ht} \sum_{n=1}^{\infty} (a_n \cos q_n t + b_n \sin q_n t) \sin \frac{n\pi x}{l},$$

$$q_n = \sqrt{\frac{n^2 a^2 \pi^2}{l^2} - h^2},$$

with

$$a_n = \frac{2}{l} \int_0^l \varphi_0(x) \sin \frac{n\pi x}{l} dx,$$

$$b_n = \frac{h}{q_n} a_n + \frac{2}{l q_n} \int_0^l \varphi_1(x) \sin \frac{n\pi x}{l} dx.$$

Hint: Apply the method of the separation of variables to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} + 2h \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions

$$u(0, t) = 0, \quad u(l, t) = 0;$$

$$u(x, 0) = \varphi_0(x), \quad \frac{\partial u(x, 0)}{\partial t} = \varphi_1(x).$$

where h is a small positive constant.

72. $$u(x, t) = \frac{A \sin\left(\frac{\omega}{a} x\right) \sin \omega t}{\sin \frac{\omega}{a} l} +$$

$$+\frac{2A\omega a}{l}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\omega^2-\left(\frac{n\pi a}{l}\right)^2}\sin\frac{n\pi at}{l}\sin\frac{n\pi x}{l}.$$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2} \tag{*}$$

subject to the conditions

$$u(0, t)=0,\quad u(l, t)=A\sin\omega t;$$

$$u(x, 0)=0,\quad \frac{\partial u(x, 0)}{\partial t}=0$$

The solution can be found by assuming that it has the form $u=v+w$ where w is a solution of the equation (*) which satisfies the conditions $w(0, t)=0$; $w(l, t)=A\,.\sin\omega t$; and where v is a solution of the equation (*) with

$$v(0, t)=0,\quad v(l, t)=0;$$

$$v(x, 0)=-w(x, 0),\quad \frac{\partial v(x, 0)}{\partial t}=-\frac{\partial w(x, 0)}{\partial t}.$$

73. $u(x, t)=$

$$=\frac{Q}{E}x-\frac{8Ql}{\pi^2E}\sum_{n=0}^{\infty}(-1)^n\frac{\cos\frac{(2n+1)\pi at}{2l}\sin\frac{(2n+1)\pi x}{2l}}{(2n+1)^2}.$$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$$

subject to the conditions

$$u(0, t)=0,\quad \frac{\partial u(l, t)}{\partial x}=\frac{Q}{E};$$

$$u(x, 0)=0,\quad \frac{\partial u(x, 0)}{\partial t}=0$$

See also the hint for problem **72.**

74. $$u(x,t) = \frac{aA}{E\omega}\,\frac{\sin\left(\dfrac{\omega}{a}x\right)\sin\omega t}{\cos\dfrac{\omega l}{a}} +$$

$$+\frac{2a\omega A}{El}\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{k_n}\cdot\frac{\sin k_n x}{\omega^2 - k_n^2 a^2}\sin ak_n t,$$

with $k_n = \dfrac{(2n+1)\pi}{2l}$ and $\omega \neq ak_n$.

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\,\frac{\partial^2 u}{\partial x^2}$$

subject to the conditions

$$u(0,\ t) = 0, \quad \frac{\partial u(l,t)}{\partial x} = \frac{A}{E}\sin\omega t;$$

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0.$$

See also the hint for problem **72.**

75. $$u_1 = \frac{A\sin\dfrac{\omega x}{a_1}\sin\omega t}{Q}; \quad u_2 = \frac{P}{Q},$$

with

$$P = A\left(\sin\frac{\omega c}{a_1}\sin\frac{\omega c}{a_2} + \frac{E_1 a_2}{E_2 a_1}\cos\frac{\omega c}{a_1}\cos\frac{\omega c}{a_2}\right)\sin\frac{\omega x}{a_2}\sin\omega t +$$

$$+ A\left(\sin\frac{\omega c}{a_1}\cos\frac{\omega c}{a_2} - \frac{E_1 a_2}{E_2 a_1}\sin\frac{\omega c}{a_2}\cos\frac{\omega c}{a_1}\right)\cos\frac{\omega x}{a_2}\sin\omega t;$$

$$Q = \left(\sin\frac{\omega c}{a_1}\cos\frac{\omega c}{a_2} - \frac{E_1 a_2}{E_2 a_1}\sin\frac{\omega c}{a_2}\cos\frac{\omega c}{a_1}\right)\cos\frac{\omega l}{a_2} +$$

$$+\left(\sin\frac{\omega c}{a_1}\sin\frac{\omega c}{a_2} + \frac{E_1 a_2}{E_2 a_1}\cos\frac{\omega c}{a_1}\cos\frac{\omega c}{a_2}\right)\sin\frac{\omega l}{a_2}.$$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u_1}{\partial t^2} = a_1^2 \frac{\partial^2 u_1}{\partial x^2} \quad (0 < x < c);$$

$$\frac{\partial^2 u_2}{\partial t_2} = a_2^2 \frac{\partial^2 u_2}{\partial x^2} \quad (c < x < l), \quad \text{with } a_i^2 = \frac{E_i}{\rho_i},$$

subject to the conditions

$$u_1(0, t) = 0, \quad u_2(l, t) = A \sin \omega t;$$

$$u_1(c, t) = u_2(c, t), \quad E_1 \frac{\partial u_1(c, t)}{\partial x} = E_2 \frac{\partial u_2(c, t)}{\partial x}.$$

A solution can be found by assuming that there are solutions of the form

$$u_1(x, t) = X_1(x) \sin \omega t, \quad u_2(x, t) = X_2(x) \sin \omega t.$$

76. $$u(x, t) = \frac{gx(2l - x)}{2a^2} - $$

$$- \frac{16gl^2}{\pi^3 a^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \cos \frac{(2n+1)\pi a t}{2l} \sin \frac{(2n+1)\pi x}{2l},$$

where g is the constant of gravitation.
Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + g \tag{*}$$

subject to the conditions

$$u(0, t) = 0, \quad \frac{\partial u(l, t)}{\partial x} = 0;$$

$$u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = 0.$$

Assume that the solution has the form $u = v + w$, where v is a solution of the inhomogeneous equation (*)
with

$$v(0, t) = 0, \quad \frac{\partial v(l, t)}{\partial x} = 0$$

and which has the form $Ax^2 + Bx + C$; w is a solution of the homogeneous equation, which satisfies the condition

$$w(0, t) = 0, \quad \frac{\partial w(l, t)}{\partial x} = 0;$$

$$w(x, 0) = -v(x, 0), \quad \frac{\partial w(x, 0)}{\partial t} = -\frac{\partial v(x, 0)}{\partial t}.$$

77. $$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t) \sin \frac{n\pi x}{l} + \sum_{n=1}^{\infty} \frac{f_n}{\omega_n(\omega^2 - \omega_n^2)} (\omega \sin \omega_n t - \omega_n \sin \omega t) \sin \frac{n\pi x}{l},$$

with

$$a_n = \frac{2}{l} \int_0^l \varphi_0(x) \sin \frac{n\pi x}{l} dx, \quad b_n = \frac{2}{n\pi a} \int_0^l \varphi_1(x) \sin \frac{n\pi x}{l} dx,$$

$$f_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx, \quad \omega_n = \frac{an\pi}{l}.$$

Resonance will occur if the frequency ω of the applied external force corresponds with one of the characteristic frequencies

$$\omega_{n_1} = \frac{an_1\pi}{l}$$

of the string.

When resonance occurs the solution has the form

$$u(x, t) = \frac{f_{n_1}}{2\omega_{n_1}^3} (\sin \omega_{n_1} t - t\omega_{n_1} \cos \omega_{n_1} t) \sin \frac{n_1 \pi x}{l} + \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t) \sin \frac{n\pi x}{l} + \sum_{n=1}^{\infty}{}' \frac{f_n}{\omega_n(\omega^2 - \omega_n^2)} (\omega \sin \omega_n t - \omega_n \sin \omega t) \sin \frac{n\pi x}{l},$$

where the prime indicates that the summation is only over terms with $n \neq n_1$.

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x) \sin \omega t$$

subject to the conditions

$$u(0, t) = 0, \quad u(l, t) = 0,$$

$$u(x, 0) = \varphi_0(x), \quad \frac{\partial u(x, 0)}{\partial t} = \varphi_1(x) \quad (0 \leqslant x \leqslant l).$$

78. $$u(x, t) = \frac{b}{a^2}\left(\frac{x}{l} \mathfrak{Sin}\, l - \mathfrak{Sin}\, x\right) + \frac{2b}{a^2\pi} \sum_{n-1}^{\infty} \frac{(-1)^n}{n} \cos\frac{n\pi at}{l} \sin\frac{n\pi x}{l} - \frac{2b\pi\, \mathfrak{Sin}\, l}{a^2} \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2\pi^2 + l^2} \cos\frac{n\pi at}{l} \sin\frac{n\pi x}{l}.$$

Hint: A solution can be found by assuming that it has the form $u(x, t) = v(x) + w(x, t)$, where $v(x)$ is a solution of the ordinary differential equation

$$a^2 \cdot v''(x) + b \cdot \mathfrak{Sin}\, x = 0$$

with boundary conditions $v(0) = v(1) = 0$; w is a solution of the equation

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2}$$

with conditions

$$w(0, t) = 0, \quad w(l, t) = 0,$$

$$w(x, 0) = -v(x, 0), \quad \frac{\partial w(x, 0)}{\partial t} = 0.$$

79. $$u(x, t) = -\frac{bx}{12}(x^3 - 2x^2 l + l^3) + \frac{8l^4}{\pi^5} \sum_{n=0}^{\infty} \frac{\cos\frac{(2n+1)\pi at}{l} \sin\frac{(2n+1)\pi x}{l}}{(2n+1)^5}.$$

80. $$u(x, t) = A\,\frac{\mathfrak{Sin}\, b(l - x)}{\mathfrak{Sin}\, bl} -$$

$$- 2A\, e^{-\mu t} \sum_{k=1}^{\infty} \frac{2k\pi}{b^2l^2 + k^2\pi^2} \left(\mathfrak{Cof}\, n_k t + \frac{\mu}{n_k}\, \mathfrak{Sin}\, n_k t\right) \sin \frac{k\pi x}{l},$$

with

$$\mu = \frac{h}{a^2}, \quad n_k = \frac{1}{a^2 l} \sqrt{h^2l^2 - a^2(b^2l^2 + k^2\pi^2)}.$$

81. Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right) + \omega^2 u \text{ with } a = \sqrt{g}$$

subject to the conditions

$$u(0, t) \text{ is bounded; } u(l, t) = 0$$

$$u(x, 0) = f(x), \quad \frac{\partial u(x, 0)}{\partial t} = F(x) \quad (0 \leqslant x \leqslant l).$$

82. $$u(r, t) = 8A \sum_{n=1}^{\infty} \frac{J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n^3 J_1(\mu_n)} \cos \frac{a\mu_n t}{R},$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation $J_0(\mu) = 0$. Hint: Apply the method of separation of variables to the integration of the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2},$$

with conditions:

$$u(0, t) \text{ is bounded; } u(R, t) = 0;$$

$$u(r, 0) = A\left(1 - \frac{r^2}{R^2}\right), \quad \frac{\partial u(r, 0)}{\partial t} = 0, \quad \text{with } A = \text{const.}$$

For the determination of the coëfficients in the series expansion the following formulas are useful.

$$\int_0^x t\mathrm{J}_0(t)\,dt = x\mathrm{J}_1(x),$$

$$\int_0^x t^3\mathrm{J}_0(t)\,dt = 2x^2\mathrm{J}_0(x) + (x^3 - 4x)\mathrm{J}_1(x).$$

83. $$u(r, t) = \frac{2}{R^2}\sum_{n=1}^{\infty} \mathrm{e}^{-ht}\left(\cos q_n t + \frac{h}{q_n}\sin q_n t\right)\cdot$$

$$\cdot\frac{\mathrm{J}_0\left(\frac{\mu_n r}{R}\right)}{\mathrm{J}_1^2(\mu_n)}\int_0^R \rho\varphi(\rho)\mathrm{J}_0\left(\frac{\mu_n\rho}{R}\right)d\rho,$$

with $q_n = \sqrt{\frac{a^2\mu_n^2}{R^2} - h^2}$ and where $\mu_1, \mu_2, \ldots$ are the positive roots of $\mathrm{J}_0(\mu) = 0$.

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} + 2h\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

(h is a small positive constant)
subject to the conditions:

$$u(0, t) \text{ is bounded, } u(R, t) = 0;$$

$$u(r, 0) = \varphi(r), \quad \frac{\partial u(r, 0)}{\partial t} = 0.$$

84. $u(x, t) =$

$$= \frac{1}{l}\sum_{n=1}^{\infty}\frac{\mathrm{J}_0\left(\mu_n\sqrt{\frac{x}{l}}\right)}{\mathrm{J}_1^2(\mu_n)}\cos\frac{\mu_n at}{2\sqrt{l}}\int_0^l f(x)\mathrm{J}_0\left(\mu_n\sqrt{\frac{x}{l}}\right)dx,$$

where $\mu_1, \mu_2, \ldots$ are the positive roots of $\mathrm{J}_0(\mu) = 0$.

Hint: Integrate the equation for the small vibrations of the string

subject to the conditions

$$u(0, t) \text{ is bounded}, \quad u(l, t) = 0;$$

$$u(x, 0) = f(x), \quad \frac{\partial u(x, 0)}{\partial t} = 0.$$

85. $$u(r, t) = \frac{2}{R^2} \int_0^R [f(\rho) + tF(\rho)]\rho d\rho +$$

$$+ \sum_{n=1}^{\infty} \left(a_n \cos \frac{\mu_n at}{R} + b_n \sin \frac{\mu_n at}{R} \right) J_0\left(\frac{\mu_n r}{R}\right);$$

$$a_n = \frac{2}{R^2 J_0^2(\mu_n)} \int_0^R \rho f(\rho) J_0\left(\frac{\mu_n \rho}{R}\right) d\rho,$$

$$b_n = \frac{2}{aR\mu_n J_0^2(\mu_n)} \int_0^R \rho F(\rho) J_0\left(\frac{\mu_n \rho}{R}\right) d\rho;$$

where $\mu_1, \mu_2, \ldots$ are the positive roots of $J_1(\mu) = 0$.
Hint: Take the axis of the tube as z-axis and transform the equation for the vibration of the gas to cylindrical coördinates r, φ, z. Find now the solution of the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}$$

which satisfies the conditions:

$$u(0, t) \text{ is bounded}, \quad \frac{\partial u(R, t)}{\partial r} = 0;$$

$$u(r, 0) = f(r), \quad \frac{\partial u(r, 0)}{\partial t} = F(r), \quad 0 < r < R.$$

86. $$u(x, t) = \sum_{n=1}^{\infty} [a_n \cos \sqrt{2n(2n-1)}at +$$

$$+ b_n \sin \sqrt{2n(2n-1)}at] P_{2n-1}\left(\frac{x}{l}\right),$$

with

$$a_n = \frac{4n-1}{l}\int_0^l f(x)P_{2n-1}\left(\frac{x}{l}\right)dx;$$

$$b_n = \frac{4n-1}{\sqrt{2n(2n-1)}\cdot al}\int_0^l F(x)P_{2n-1}\left(\frac{x}{l}\right)dx.$$

where

$$P_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}[(x^2-1)^k]$$

are the Legendre polynomials.
Hint: The problem can be reduced to the integration of the equation

$$(l^2 - x^2)\frac{\partial^2 u}{\partial x^2} - 2x\frac{\partial u}{\partial x} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}, \quad a = \frac{\omega}{\sqrt{2}}$$

subject to the conditions

$$u(0, t) = 0; \quad u(l, t) \text{ is bounded};$$

$$u(x, 0) = f(x), \quad \frac{\partial u(x, 0)}{\partial t} = F(x).$$

87. Hint: The problem can be reduced to the integration of the following equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{P}{T}.$$

subject to the conditions

$$u(0, t) \text{ is bounded}; \; u(R, t) = 0;$$

$$u(x, 0) = 0, \quad \frac{\partial u(r, 0)}{\partial t} = 0.$$

To determine the coëfficients in the series expansion compare with the hint for problem **82.**

88. $$u(r,t) = \frac{a^2 P_0}{T\omega^2}\left[\frac{J_0\left(\frac{\omega r}{a}\right)}{J_0\left(\frac{\omega}{a}R\right)} - 1\right]\sin\omega t -$$

$$-\frac{2aP_0\omega R^3}{T}\sum_{n=1}^{\infty}\frac{\sin\frac{\mu_n at}{R}\, J_0\left(\frac{\mu_n r}{R}\right)}{\mu_n^2(\omega^2R^2 - a^2\mu_n^2)J_0'(\mu_n)},$$

$\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation $J_0(\mu) = 0$. Hint: The problem can be reduced to solving the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{P_0 \sin\omega t}{T} \qquad (*)$$

where the following conditions are imposed:

$u(0, t)$ is bounded; $u(R, t) = 0$;

$$u(r, 0) = 0, \quad \frac{\partial u(r, 0)}{\partial t} = 0.$$

The solution of the equation can be found using the form $u=v+w$, where v is a solution of the inhomogeneous equation (*) and satisfies the conditions:

$v(0, t)$ is bounded; $v(R, t) = 0$.

Choose for v the form $B(r)\cdot\sin\omega t$; w is the solution of the corresponding homogeneous equation and satisfies the conditions

$w(0, t)$ is bounded; $w(R, t) = 0$;

$$w(r, 0) = -v(r, 0), \quad \frac{\partial w(r, 0)}{\partial t} = -\frac{\partial v(r, 0)}{\partial t}.$$

See the hint for problem **82** for the calculation of the coëfficients of the series expansion.

89. $$u(x,y,t) = \frac{64Ab^4}{\pi^6}\sum_{n,m=0}^{\infty}\frac{\sin\frac{(2n+1)\pi x}{b}\sin\frac{(2m+1)\pi y}{b}}{(2n+1)^3(2m+1)^3}\times$$

$$\times\cos\sqrt{(2n+1)^2+(2m+1)^2}\,\frac{a\pi t}{b}.$$

Hint: The problem is equivalent to solving the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the conditions

$$u|_{x=0} = u|_{x=b} = u|_{y=0} = u|_{y=b} = 0,$$

$$u|_{t=0} = Axy(b-x)(b-y), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

90. $$u(x, y, t) = \frac{4A}{a\pi ml} \sum_{k,\nu=1}^{\infty} \frac{\psi_{k\nu}\left(\frac{l}{2}, \frac{m}{2}\right)}{\mu_{k\nu}} \psi_{k\nu}(x, y) \sin \mu_{k\nu}\pi a t,$$

with $\psi_{k\nu}(x, y) = \sin \frac{k\pi x}{l} \sin \frac{\nu\pi y}{m}, \quad \mu_{k\nu} = \sqrt{\left(\frac{k}{l}\right)^2 + \left(\frac{\nu}{m}\right)^2}.$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the following conditions

$$u|_{x=0} = u|_{x=l} = u|_{y=0} = u|_{y=m} = 0.$$

$u(x, y, 0) = 0, \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$, but for a sufficiently small environment of the point $\left(\frac{l}{2}, \frac{m}{2}\right)$.

91. $$u(r, \theta, \varphi, t) = e^{im\varphi} \sum_{j,\nu=1}^{\infty} (a_{\nu j} \cos k_j vt + b_{\nu j} \sin k_j vt) v_{\nu j}(r, \theta),$$

with

$$v_{\nu j}(r, \theta) = \frac{Y_{l_\nu+\frac{1}{2}}(k_j b) J_{l_\nu+\frac{1}{2}}(k_j r) - J_{l_\nu+\frac{1}{2}}(k_j b) Y_{l_\nu+\frac{1}{2}}(k_j r)}{\sqrt{r}} \times$$

$$\times P_{l_\nu,m}(\cos\theta), \quad P_{l_\nu,m}(x) = (1-x)^{m/2} \frac{d^m P_{l_\nu}(x)}{dx^m},$$

$P_{l_\nu}(x)$ are the Legendre polynomials, and $l_1, l_2, l_3, \ldots$ are the real

roots of the equation

$$P_{l,m}(\cos \alpha) = 0$$

and $k_1, k_2, k_3, \ldots$ are the real roots of the equation

$$Y_{l+\frac{1}{2}}(kb)J_{l+\frac{1}{2}}(ka) - J_{l+\frac{1}{2}}(kb)Y_{l+\frac{1}{2}}(ka) = 0,$$

$$a_{\nu j} = \frac{1}{A_{\nu j}^2} \int\limits_a^b \int\limits_0^\alpha \varphi(r, \theta) v_{\nu j}(r, \theta) r^2 \sin \theta \, dr \, d\theta,$$

$$b_{\nu j} = \frac{1}{k_j A_{\nu j}^2} \int\limits_a^b \int\limits_0^\alpha \varphi_1(r, \theta) v_{\nu j}(r, \theta) r^2 \sin \theta \, dr \, d\theta,$$

$$A_{\nu j}^2 = \int\limits_a^b \int\limits_0^\alpha v_{\nu j}^2(r, \theta) r^2 \sin \theta \, dr \, d\theta.$$

Hint: Introduce spherical coördinates r, θ, φ and look for a solution of the form

$$u(r, \theta, \varphi, t) = T(t) v(r, \theta, \varphi).$$

Substitute this in the wave equation and separate the variables. There follows:

$$T''(t) + k^2 T(t) = 0,$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{2}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial v}{\partial \theta}\right) + \\ + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v}{\partial \varphi^2} + k^2 v = 0.$$

We write the solution of this equation in the following form:

$$v(r, \theta, \varphi) = e^{im\varphi} R(r) P(\theta).$$

92. $$u(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n^2\pi^2 b}{l^2} t + B_n \sin \frac{n^2\pi^2 b}{l^2} t\right) \sin \frac{n\pi x}{l};$$

$$A_n = \frac{2}{l} \int\limits_0^l \varphi_0(x) \sin \frac{n\pi x}{l} \, dx, \quad B_n = \frac{2l}{n^2\pi^2 b} \int\limits_0^l \varphi_1(x) \sin \frac{n\pi x}{l} \, dx.$$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} + b^2 \frac{\partial^4 u}{\partial x^4} = 0$$

subject to the following conditions

$$u = 0, \quad \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for } x = 0,\ x = l \text{ and arbitrary } t > 0;$$

$$u(x, 0) = \varphi_0(x), \quad \frac{\partial u(x, 0)}{\partial t} = \varphi_1(x).$$

93. Hint: The problem can be reduced to the integration of the telegraphist equation

$$-\frac{\partial V}{\partial x} = RI + L\frac{\partial I}{\partial t}, \qquad -\frac{\partial I}{\partial x} = GV + C\frac{\partial V}{\partial t}$$

with the initial conditions

$$V(x, 0) = E, \quad I(x, 0) = 0$$

and the boundary conditions

$$V(0, t) = 0, \quad I(l, t) = 0.$$

94. $$I = \frac{2E}{lL} e^{-(Rt/2L)} \sum_{n=0}^{\infty} \frac{\sin \nu_n t}{\nu_n} \cos \frac{(2n+1)\pi x}{2l},$$

$$V = E - \frac{4E}{\pi} e^{-(Rt/2L)} \sum_{n=0}^{\infty} \frac{\cos \nu_n t}{2n+1} \sin \frac{(2n+1)\pi x}{2l} -$$

$$- \frac{2ER}{\pi L} e^{-(Rt/2L)} \sum_{n=0}^{\infty} \frac{\sin \nu_n t}{\nu_n(2n+1)} \sin \frac{(2n+1)\pi x}{2l},$$

with

$$\nu_n = \sqrt{\frac{(2n+1)^2\pi^2}{4l^2CL} - \frac{R^2}{4L^2}}.$$

Hint: The problem can be reduced to the integration of the first order system

$$-\frac{\partial V}{\partial x} = RI + L\frac{\partial I}{\partial t}, \quad -\frac{\partial I}{\partial x} = C\frac{\partial V}{\partial t}$$

with boundary conditions

$$I(l, t) = 0, \quad V(0, t) = E$$

and initial conditions

$$I(x, 0) = 0, \quad V(x, 0) = 0.$$

In order to solve this, put:

$$I(x, t) = \sum_{n=0}^{\infty} \tau_n(t) \cos \frac{(2n+1)\pi x}{2l},$$

$$V(x, t) = E + \sum_{n=0}^{\infty} T_n(t) \sin \frac{(2n+1)\pi x}{2l}.$$

95. $$u(x, t) = \frac{4l}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} e^{-[(2n+1)^2\pi^2a^2/l^2]t} \cdot \sin \frac{(2n+1)\pi x}{l}.$$

96. $$u(x, t) = \frac{8c}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} e^{-[(2n+1)^2\pi^2a^2/l^2]t} \times$$
$$\times \sin \frac{(2n+1)\pi x}{l}.$$

97. $$u(r, t) = \frac{2}{Rr} \sum_{n=1}^{\infty} e^{-(n^2\pi^2a^2/R^2)t} \sin \frac{n\pi r}{R} \cdot \int_0^R \rho f(\rho) \sin \frac{n\pi\rho}{R}\, d\rho.$$

Hint: The problem can be reduced to solving the equation

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial r^2} \quad \text{with } v = ru, \quad a = \sqrt{\frac{k}{c\rho}}$$

subject to the following conditions

$$v(0, t) = 0, \quad v(R, t) = 0, \quad v(r, 0) = rf(r).$$

98. $$u(x, t) =$$
$$= \frac{2}{l} \sum_{n=1}^{\infty} \frac{p^2 + \mu_n^2}{p(p+1) + \mu_n^2} e^{-(\mu_n^2a^2t/l^2)} \cdot \sin \frac{\mu_n x}{l} \int_0^l f(x) \sin \frac{\mu_n x}{l}\, dx,$$

$\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation

$$\operatorname{tg} \mu = -\frac{\mu}{p}, \quad p = HL > 0.$$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

with the conditions

$$u(0, t) = 0, \quad \frac{\partial u}{\partial x} + Hu|_{x=l} = 0, \quad H = \frac{h}{k} > 0, \quad u(x, 0) = f(x).$$

99. $$u(x, t) = \frac{2}{l} \sum_{n=1}^{\infty} A_n e^{-(\mu_n^2 a^2/l^2)t} \frac{\mu_n \cos \frac{\mu_n x}{l} + p \sin \frac{\mu_n x}{l}}{p(p+2) + \mu_n^2}$$

with

$$A_n = \int_0^l f(z) \left(\mu_n \cos \frac{\mu_n z}{l} + p \sin \frac{\mu_n z}{l} \right) dz;$$

$\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation

$$2 \operatorname{ctg} \mu = \frac{\mu}{p} - \frac{p}{\mu}, \quad \text{with } p = \frac{h}{k} l.$$

Hint: The boundary conditions have the form

$$\frac{\partial u}{\partial x} - Hu|_{x=0} = 0, \quad \frac{\partial u}{\partial x} + Hu|_{x=l} = 0.$$

100. $$u(r, t) = \frac{2}{Rr} \sum_{n=1}^{\infty} \frac{p^2 + \mu_n^2}{p(p+1) + \mu_n^2} e^{-(\mu_n^2 a^2 t/R^2)} \sin \frac{\mu_n r}{R} \times$$
$$\times \int_0^R \rho f(\rho) \sin \frac{\mu_n \rho}{R} \, d\rho,$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of

$$\operatorname{tg} \mu = -\frac{\mu}{p}, \quad p = HR - 1 > -1.$$

Hint: The second boundary condition in problem **97** must be

replaced by the following condition

$$\frac{\partial v}{\partial r} + \left(H - \frac{1}{R}\right) v \Big|_{r=R} = 0, \quad H = \frac{h}{k}.$$

101. $u(x, t) = 4u_0 \sum_{n=1}^{\infty} \frac{\sin \mu_n R - \mu_n R \cos \mu_n R}{\mu_n(4\mu_n R - \sin 4\mu_n R)} \cdot \frac{\sin \mu_n r}{r} e^{-\mu_n^2 a^2 t}$,

with

$$\operatorname{tg} 2\mu_n R = \frac{2\mu_n R}{1 - 2hR}.$$

Hint: The problem can be reduced to solving the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} \right)$$

where the following conditions must be satisfied:

$$u(0, t) \text{ is bounded}, \quad \frac{\partial u}{\partial r} + hu|_{r=2R} = 0;$$

$$u(r, 0) = \begin{cases} u_0 & \text{for } 0 \leqslant r \leqslant R, \\ 0 & \text{for } R \leqslant r \leqslant 2R. \end{cases}$$

102. $u(x, t) = u_0 + \sum_{n=0}^{\infty} a_n e^{-a^2 \lambda_n^2 t} \cos \lambda_n x$,

with

$$a_n = \frac{2}{l} \int_0^t \varphi(x) \cos \frac{(2n+1)\pi x}{2l} dx - \frac{4u_0}{\pi(2n+1)},$$

$$\lambda_n = \frac{(2n+1)\pi}{2l}.$$

Hint: One can reduce the problem to the integration of the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions

$$\frac{\partial u(0, t)}{\partial x} = 0, \quad u(l, t) = u_0, \quad u(x, 0) = \varphi(x).$$

The solution is found by means of the assumption

$$u(x, t) = u_0 + v(x, t),$$

where $v(x, t)$ is to be found.

103. $$u(x, t) = At\left(1 - \frac{x}{l}\right) - \frac{l^2 A}{6a^2}\left[\left(\frac{x}{l}\right)^3 - 3\left(\frac{x}{l}\right)^2 + 2\left(\frac{x}{l}\right)\right] +$$

$$+ \frac{2l^2 A}{\pi^3 a^2} \sum_{n=1}^{\infty} \frac{e^{-(n^2\pi^2 a^2 t/l^2)}}{n^3} \sin\frac{n\pi x}{l}.$$

Hint: Find a solution in the form of a sum of two terms

$$u(x, t) = u_1(x, t) + u_2(x, t),$$

where $u_1(x, t)$ is a solution of the equation

$$\frac{\partial u_1}{\partial t} = a^2 \frac{\partial^2 u_1}{\partial x^2}$$

which satisfies the following conditions

$$u_1(0, t) = At, \quad u_1(l, t) = 0,$$

and u_2 is a solution of the same equation with the conditions

$$u_2(0, t) = 0, \quad u_2(l, t) = 0, \quad u_2(x, 0) = -u_1(x, 0).$$

104. $$u(x, t) = A\left(1 - \frac{x}{l}\right)\sin \omega t -$$

$$- \frac{2\omega A}{\pi} \sum_{n=1}^{\infty} \frac{e^{-(n\pi a/l)^2 t} \sin\dfrac{n\pi x}{l}}{n} \int_0^l e^{(n\pi a/l)^2 \tau} \cos \omega\tau \, d\tau.$$

Hint: To solve this problem, put

$$u(x, t) = A\left(1 - \frac{x}{l}\right)\sin \omega t + v(x, t),$$

where v is a solution of the equation

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} + A\omega\left(\frac{x}{l} - 1\right)\cos \omega t$$

and satisfies the following conditions

$$v(0, t) = 0, \quad v(l, t) = 0, \quad v(x, 0) = 0.$$

105. $$u(x, t) = A(1 - e^{-\alpha t})\left(1 - \frac{Hx}{1+p}\right) + $$
$$+ 2A\alpha \sum_{n=1}^{\infty} \frac{1}{\mu_n} \cdot \frac{p^2 + \mu_n}{p(p+1) + \mu_n^2} \cdot \frac{l^2}{a^2\mu_n^2 - \alpha l^2} \times$$
$$\times [e^{-(a\mu_n/l)^2 t} - e^{-\alpha t}],$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of

$$\operatorname{tg} \mu = -\frac{\mu}{p} \quad (p = Hl > 0).$$

Hint: If $u(x, t)$ is the solution, put:

$$u(x, t) = A(1 - e^{-\alpha t})\left(1 - \frac{Hx}{1+p}\right) + \omega,$$

where ω is a solution of the equation

$$\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2} - A\alpha e^{-\alpha t}\left(1 - \frac{Hx}{1+p}\right)$$

which satisfies the conditions

$$w(0, t) = 0, \quad \frac{\partial w}{\partial x} + Hw|_{x=l} = 0, \quad w|_{t=0} = 0.$$

106. $$u(r, t) = bt - \frac{b}{6a^2}\left[R^2\left(1 + \frac{2}{HR}\right) - r^2\right] +$$
$$+ \frac{2bR^3}{a^2 r} \sum_{n=1}^{\infty} \frac{\sin \mu_n - \mu_n \cos \mu_n}{\mu_n^3(\mu_n - \sin \mu_n \cos \mu_n)} e^{-(\mu_n^2 a^2 t/R^2)} \sin \frac{\mu_n r}{R},$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation

$$\operatorname{tg} \mu = -\frac{\mu}{p}, \quad p = HR - 1 > 0.$$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial r^2} \quad (v = ru) \tag{*}$$

subject to the conditions

$$v(0, t) = 0, \quad \frac{\partial v}{\partial r} + \left(H - \frac{1}{R}\right) v \Big|_{r=R} = RHbt \qquad (**)$$

$$v(r, 0) = 0.$$

By means of the assumption $v = v_1 + v_2$ the solution is obtained. v_1 is a solution of the equation (*) and satisfies the conditions (**); v_2 is a solution of the same equation, however with the following conditions

$$v_2(0, t) = 0, \quad \frac{\partial v_2}{\partial r} + \left(H - \frac{1}{R}\right) v_2 \Big|_{r=R} = 0,$$

$$v_2(r, 0) = -v_1(r, 0).$$

107. $$u(x, t) = \frac{a^2 q}{kR}\left(t - \frac{R^2 - 3x^2}{6a^2}\right) - $$

$$- \frac{2qR}{k\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{R} \, e^{-(n^2\pi^2 a^2/R^2)t};$$

where k is the coëfficient of heat conduction.

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the following conditions

$$k \frac{\partial u}{\partial x} + q \Big|_{x=-R} = 0, \quad -k \frac{\partial u}{\partial x} + q \Big|_{x=R} = 0, \quad \frac{\partial u(0, t)}{\partial x} = 0,$$

$$u(x, 0) = 0.$$

108. $$u(x, t) = \frac{u_1 \operatorname{Sin} \frac{b}{a} x - u_0 \operatorname{Sin} \frac{b}{a}(x - l)}{\operatorname{Sin} \frac{b}{a} l} + $$

$$+ \frac{2\pi}{l^2} \sum_{n=1}^{\infty} n \frac{(-1)^n u_1 - u_0}{\lambda_n^2} e^{-a^2\lambda_n^2 t} \sin \frac{n\pi x}{l} + $$

$$+ \frac{2}{l} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} e^{-a^2\lambda_n^2 t} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx,$$

with

$$\lambda_n^2 = \frac{n^2\pi^2}{l^2} + \frac{b^2}{a^2}.$$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} = b^2 u \quad \left(a^2 = \frac{k}{c\rho}, \quad b^2 = \frac{hp}{c\rho\omega}\right) \tag{*}$$

subject to the conditions

$$u(0, t) = u_0, \quad u(l, t) = u_1, \quad u(x, 0) = f(x).$$

Here k, h are the coëfficients of heat conduction, resp. heat exchange; c, ρ the specific heat resp. linear mass density of the rod; ω, p the cross-seçtion resp. perimeter of the rod.
The solution follows from the asumption $u = v + w$, where v is a solution of the equation

$$a^2 \frac{d^2 v}{dx^2} - b^2 v = 0$$

and satisfies the conditions

$$v(0) = u_0, \quad v(l) - u_1,$$

and w is a solution of the equation (*) and satisfies the conditions

$$w(0, t) = 0, \quad w(l, t) = 0, \quad w(x, 0) = f(x) - v(x).$$

109. $$u(x, t) = u_0 \frac{b\,\mathfrak{Sin}\,\frac{b}{a}(l - x) + Ha\,\mathfrak{Sin}\,\frac{b}{a}(l - x)}{b\,\mathfrak{Cof}\,\frac{bl}{a} + Ha\,\mathfrak{Sin}\,\frac{bl}{a}} -$$

$$- 2u_0 a^2 \sum_{n=1}^{\infty} \frac{\mu_n(\mu_n^2 + H^2)}{(a^2\mu_n^2 + b^2)[l(\mu_n^2 + H^2) + H]} e^{-(a^2\mu_n^2 + b^2)t} \sin \mu_n x.$$

μ_n is obtained from the equation

$$\operatorname{tg} l\mu = -\frac{\mu}{H}.$$

Hint: The problem can be reduced to the solution of the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} - b^2 u \tag{*}$$

subject to the conditions

$$u(0, t) = u_0, \quad \left.\frac{\partial u}{\partial x} + Hu\right|_{x=l} = 0, \quad u(x, 0) = 0.$$

For the solution we put $u = v + w$, where v is a solution of the equation

$$a^2 \frac{d^2 v}{dx^2} - b^2 v = 0$$

which satisfies the following conditions

$$v(0) = u_0, \quad \frac{dv(l)}{dx} + Hv(l) = 0;$$

w is a solution of the equation (*) and satisfies the conditions

$$w(0, t) = 0, \quad \left.\frac{\partial w}{\partial x} + Hw\right|_{x=l} \quad 0, \quad w(x, 0) = -v(x).$$

110. $\dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial \theta^2} - b(u - u_0), \quad a^2 = \dfrac{k}{c\rho}, \quad b = \dfrac{hp}{\sigma k},$

Here θ is the length of arc; k, h are the coefficients of heat conduction resp. exchange; c, ρ the specific heat capacity resp. linear mass density; and σ, p the cross-section and perimeter of the rod, u_0 is the temperature of the surrounding medium.

$$u(\theta, t) = \mathrm{e}^{-bt}[a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\mathrm{e}^{-n^2 a^2 t}],$$

with

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta\, d\theta,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta\, d\theta.$$

111. $$u(x,t) = \frac{d}{b^2}\left[1 - \frac{\mathfrak{Cof}\,\frac{b}{a}\left(x - \frac{l}{2}\right)}{\mathfrak{Cof}\,\frac{bl}{2a}}\right] -$$

$$- \frac{4d}{\pi}\sum_{n=0}^{\infty} \frac{e^{-\{[((2n+1)^2\pi^2a^2)/l^2]+b^2\}t}}{(2n+1)\left[\frac{(2n+1)^2\pi^2a^2}{l^2} + b^2\right]} \sin\frac{(2n+1)\pi x}{l} +$$

$$+ \frac{u_0\,\mathfrak{Sin}\,\frac{b}{a}(l-x) + u_1\,\mathfrak{Sin}\,\frac{bx}{a}}{\mathfrak{Sin}\,\frac{bl}{a}} +$$

$$+ 2a^2\pi \sum_{n=1}^{\infty} n\,\frac{(-1)^n u_1 - u_0}{n^2\pi^2a^2 + b^2l^2}\, e^{-[(n^2\pi^2a^2+b^2l^2)/l^2]t} \sin\frac{n\pi x}{l}.$$

Hint: The problem can be reduced to solving the equation

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} - b^2u + d \left(a^2 = \frac{k}{c\rho},\ b^2 = \frac{hp}{c\rho\omega},\ d = \frac{I^2}{c\rho\omega^2\sigma}\right)$$

subject to the conditions

$$u(0,t) = u_0,\quad u(l,t) = u_1,\quad u(x,0) = 0.$$

Here I is the current intensity, σ the electrical conductivity and ω the cross-section of the conductor.
The solution can be obtained if we write $u = v + w$, where v is a solution of the equation

$$a^2\frac{d^2v}{dx^2} - b^2v + d = 0$$

which satisfies the following conditions

$$v(0) = u_0,\quad v(l) = u_1;$$

w is a solution of the equation

$$\frac{\partial w}{\partial t} = a^2\frac{\partial^2 w}{\partial x^2} - b^2 w$$

and satisfies the conditions

$$w(0, t) = 0, \quad w(l, t) = 0, \quad w(x, 0) = -v(x).$$

112. $$u(r, t) = u_0 \left[1 + 2 \sum_{n=1}^{\infty} \frac{J_0\left(\frac{\mu_n}{R} r\right)}{\mu_n J_0'(\mu_n)} e^{-(\mu_n^2 a^2/R^2)t} \right],$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the following equation $J_0(\mu) = 0$.

Hint: One can reduce the problem to the integration of the equation

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right)$$

where the following conditions we have to be satisfied:

$u(0, t)$ is bounded $u(R, t) = u_0, \quad u(r, 0) = 0.$

113. $u(r, t) =$

$$= \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{\mu_n^2}{\mu_n^2 + H^2R^2} \times$$

$$\times\, e^{-(\mu_n^2 a^2 t/R^2)} \frac{J_0\left(\mu_n \frac{r}{R}\right)}{J_0^2(\mu_n)} \int_0^R \rho f(\rho) J_0\left(\frac{\mu_n \rho}{R}\right) d\rho,$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the following equation

$$\mu J_0'(\mu) + HRJ_0(\mu) = 0.$$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right)$$

where the following conditions have to be satisfied:

$u(0, t)$ is bounded

$$\left. \frac{\partial u}{\partial r} + Hu \right|_{r=R} = 0,$$

$$u(r, 0) = f(r).$$

114.
$$u(x, t) = \begin{cases} \sum_{j=1}^{\infty} \dfrac{k_1 a_1 u_0 A_j^2}{\mu_j} (1 - e^{-\mu_j^2 t}) \sin a_1 \mu_j x & (0 < x < \xi), \\ \sum_{j=1}^{\infty} \dfrac{k_1 a_1 u_0 A_j^2}{\mu_j} (1 - e^{-\mu_j^2 t}) \dfrac{\sin a_1 \mu \xi}{\sin a_2 \mu (l - \xi)} \sin a_2 \mu_j (l - x) \\ & (\xi < x < l). \end{cases}$$

Here the μ_j are the roots of the equation

$$a_1 k_1 \operatorname{ctg} \mu a_1 \xi + a_2 k_2 \operatorname{ctg} a_2 (l - \xi) \mu = 0,$$

and the A_j are determined by means of the normalisation

$$c_1 \rho_1 \int_0^{\xi} X_{1j}^2 dx + c_2 \rho_2 \int_{\xi}^{l} X_{2j}^2 dx = 1,$$

$$X_{1j} = A_j \sin a_1 \mu_j x,$$

$$X_{2j} = A_j \frac{\sin a_1 \mu_j \xi}{\sin a_2 \mu_j (l - \xi)} \sin a_2 \mu_j (l - x).$$

The condition for the orthogonality is

$$c_1 \rho_1 \int_0^{\xi} X_{1j} X_{1k} dx + c_2 \rho_2 \int_{\xi}^{l} X_{2j} X_{2k} dx = 0 \quad (j \neq k).$$

Hint: One can reduce the problem to the integration of the equations

$$a_1^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (0 < x < \xi),$$

$$a_2^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (\xi < x < l),$$

$$a_i^2 = \frac{c_i \rho_i}{k_i} \quad (i = 1, 2)$$

where the following conditions have to be satisfied:

$$u(0, t) = u_0, \quad u(l, t) = 0,$$

$$u(\xi - 0, t) = u(\xi + 0, t);$$

$$k_1 \frac{\partial u(\xi - 0, t)}{\partial x} = k_2 \frac{\partial u(\xi + 0, t)}{\partial x},$$

$$u(x, 0) = 0.$$

115. $$u = \frac{2}{l}\sum_{n=1}^{\infty}\left[\mathfrak{Sin}\,\frac{n\pi(m-y)}{l}\int_0^l \varphi_0(x)\sin\frac{n\pi x}{l}\,dx + \right.$$

$$\left. + \mathfrak{Sin}\,\frac{n\pi y}{l}\int_0^l \varphi_1(x)\sin\frac{n\pi x}{l}\,dx\right]\frac{\sin\frac{n\pi x}{l}}{\mathfrak{Sin}\,\frac{n\pi m}{l}} +$$

$$+ \frac{4}{lm}\sum_{\mu,\nu=1}^{\infty} e^{-a^2\pi^2[(\mu^2/l^2)+(\nu^2/m^2)]t}\sin\frac{\mu\pi x}{l}\sin\frac{\nu\pi y}{m}\times$$

$$\times\int_0^l\int_0^m\left\{f(x,y - \frac{2}{l}\sum_{n-1}^{\infty}\left[\mathfrak{Sin}\,\frac{n\pi(m-y)}{l}\int_0^l\varphi_0(\xi)\sin\frac{n\pi\xi}{l}\,d\xi + \right.\right.$$

$$\left.\left. + \mathfrak{Sin}\,\frac{n\pi y}{l}\int_0^l\varphi_1(\xi)\sin\frac{n\pi\xi}{l}\,d\xi\right]\frac{\sin\frac{n\pi x}{l}}{\mathfrak{Sin}\,\frac{n\pi m}{l}}\right\}\cdot\times$$

$$\times\sin\frac{\mu\pi x}{l}\sin\frac{\nu\pi y}{m}\,dx\,dy.$$

Hint: The solution can be obtained by writing $u = v + w$, where w is a solution of Laplace's equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

and satisfies the conditions

$$w(0, y) = 0, \quad w(l, y) = 0,$$
$$w(x, 0) = \varphi_0(x), \quad w(x, m) = \varphi_1(x)$$

v is a solution of the equation

$$\frac{\partial v}{\partial t} = a^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),$$

which satisfies the following conditions

$$v|_{x=0} = v|_{x=l} = v|_{y=0} = v|_{y=m} = 0,$$
$$v(x, y, 0) = f(x, y) - w(x, y).$$

116. $$u(r, x, t) = \sum_{k,n=1}^{\infty} A_{kn} \mathrm{J}_0\left(\mu_k \frac{r}{R}\right) \cdot$$
$$\cdot \left(\cos\frac{\nu_n x}{l} + \frac{p}{\nu_n}\sin\frac{\nu_n x}{l}\right) \mathrm{e}^{-a^2[(\nu_n{}^2/l^2)+(\mu_k{}^2/R^2)]t},$$

with

$$A_{kn} = \frac{4\mu_k^2\nu_n^2}{lR^2(\mu_k^2 + HR)[p(p+2) + \nu_n^2]} \int_0^l \int_0^R r f(r, x) \times$$
$$\times \mathrm{J}_0\left(\frac{\mu_k r}{R}\right)\left(\cos\frac{\nu_n x}{l} + \frac{p}{\nu_n}\sin\frac{\nu_n x}{l}\right) dx\,dr;$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation

$$\mu \mathrm{J}_0'(\mu) + hR\mathrm{J}_0(\mu) = 0,$$

and $\nu_1, \nu_2, \nu_3, \ldots$ the positive roots of the equation

$$2 \operatorname{ctg} \nu = \frac{\nu}{p} - \frac{p}{\nu}, \quad p = Hl > 0.$$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2}\right)$$

subject to the following conditions:

$$\frac{\partial u}{\partial x} - Hu\,|_{x=0} = 0, \quad \frac{\partial u}{\partial x} + Hu|_{x=l} = 0,$$

$$u(0, x, t) \text{ is bounded, } \frac{\partial u}{\partial r} + Hu|_{r=R} = 0,$$

$$u(r, x, 0) = f(r, x).$$

117. $$u(x, z) = \sum_{n=1}^{\infty} A_n \mathrm{e}^{-[\text{illegible}]\lambda_n t} \varphi_n(x),$$

with

$$\varphi_n(x) = x - \lambda_n\left(\frac{x^3}{3!} - \frac{6x^5}{5!}\right) + \lambda_n^2\left(\frac{x^5}{5!} - \frac{26x^7}{7!} + \frac{252x^9}{9!}\right) + \ldots,$$

The λ_n are the roots of the equation

$$1 - \tfrac{1}{4}\lambda + \frac{17}{1440}\lambda^2 - \ldots = 0,$$

$$A_n = \frac{\int_0^1 (1 - x^2)\varphi_n(x)\,dx}{\int_0^1 (1 - x^2)\varphi_n^2(x)\,dx}.$$

118. Hint: See the article of N. P. Erugin.[1]

119. $$u(x, t) = \frac{2a^2\pi}{l}\cdot\frac{\sum_{n=1}^{\infty} nA_n \sin\frac{n\pi x}{l}e^{-(a^2n^2\pi^2/l^2)t}}{A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}e^{-(a^2\pi^2n^2/l^2)t}},$$

with

$$A_0 = \frac{1}{l}\int_0^l \theta_0(x)\,dx, \quad A_n = \frac{2}{l}\int_0^l \theta_0(x)\cos\frac{n\pi x}{l}\,dx.$$

Hint: $\theta(x, t)$ is a solution of the equation of heat conduction

$$\frac{\partial\theta}{\partial t} = a^2\frac{\partial^2\theta}{\partial x^2}. \tag{1}$$

Then

$$u(x, t) = -2a^2\frac{\theta_x}{\theta} \tag{2}$$

is a solution of the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = a^2\frac{\partial^2 u}{\partial x^2}.$$

It follows from equation (2) that

$$\theta(x, t) = c(t)\,e^{-(1/2a^2)\int_0^x u(\xi,t)d\xi},$$

[1] Н. П. Еругин, Замкнутое решение параболической граничной неоднородной задачи. ПММ, т. XIV, вып. 2, 1950. (N. P. Erugin; The closed solution of a non homogeneous parabolic boundary-value problem).

and for $t = 0$

$$\theta(x, 0) = c_0 e^{-(1/2a^2)\int_0^x u(\xi,0)d\xi} = \theta_0(x). \tag{3}$$

By means of the boundary conditions in this problem and from equation (2) we obtain

$$\theta_x(0, t) = \theta_x(l, t) = 0. \tag{4}$$

In this way we get the solution of equation (1) with the conditions (3) and (4).
When this problem is solved, the solution of the posed problem is found by means of formula (2).

120. $$u(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_0^\infty f(\xi)\,[e^{-[(x-\xi)^2/4a^2t]} - e^{-[(x+\xi)^2/4a^2t]}]\, d\xi.$$

121. $$u(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_0^\infty \Big\{ f(\xi)\,[e^{-[(x-\xi)^2/4a^2t]} - e^{-[(x+\xi)^2/4a^2t]}] - 2he^{-h\xi}\int_0^\xi e^{h\mu} f(\mu)\, d\mu \Big\}\, d\xi.$$

Hint: Integrate the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions

$$\frac{\partial u}{\partial x} - hu|_{x=0} = 0, \quad u(x, 0) = f(x).$$

122. $$u_1(x, t) = \frac{u_0\sigma}{1+\sigma} \int_{-(x/2a_1\sqrt{t})}^\infty e^{-\mu^2}\, d\mu \quad (x < 0) \quad \left(\sigma = \frac{k_2 a_1}{k_1 a_2}\right),$$

$$u_2(x, t) = \frac{u_0\sigma}{1+\sigma}\left[1 + \frac{1}{\sigma}\int_0^{x/2a_2\sqrt{t}} e^{-\mu^2} d\mu\right] \quad (x > 0).$$

Hint: The problem can be reduced to the integration of the equation

$$\frac{\partial u_1}{\partial t} = a_1^2 \frac{\partial^2 u_1}{\partial x^2} \quad (x < 0); \quad \frac{\partial u_2}{\partial t} = a_2^2 \frac{\partial^2 u_2}{\partial x^2} \quad (x < 0);$$

$$a_i^2 = \frac{c_i \rho_i}{k_i} \quad (i = 1, 2)$$

where the conditions

$$u_1(0, t) = u_2(0, t), \quad k_1 \frac{\partial u_1(0, t)}{\partial x} = k_2 \frac{\partial u_2(0, t)}{\partial x},$$

$$u_1(x, 0) = 0, \quad u_2(x, 0) = u_0$$

have to be satisfied.
Here the c_i are the specific heat capacities, k_i the coëfficients of heat conduction and ρ_i the linear mass densities of the rods.

123. $$u = \frac{2}{b} \sum_{n=1}^{\infty} \left[\mathfrak{S}\mathrm{in} \frac{n\pi(a - x)}{b} \int_0^b \varphi_0(y) \sin \frac{n\pi y}{b} dy + \right.$$

$$\left. + \mathfrak{S}\mathrm{in} \frac{n\pi x}{b} \int_0^b \varphi_1(y) \sin \frac{n\pi y}{b} dy \right] \frac{\sin \frac{n\pi y}{b}}{\mathfrak{S}\mathrm{in} \frac{n\pi a}{b}} +$$

$$+ \frac{2}{a} \sum_{n=1}^{\infty} \left[\mathfrak{S}\mathrm{in} \frac{n\pi(b - y)}{a} \int_0^a \psi_0(x) \sin \frac{n\pi x}{a} dx + \right.$$

$$\left. + \mathfrak{S}\mathrm{in} \frac{n\pi y}{a} \int_0^a \psi_1(x) \sin \frac{n\pi x}{a} dx \right] \frac{\sin \frac{n\pi x}{a}}{\mathfrak{S}\mathrm{in} \frac{n\pi b}{a}}.$$

Hint: Solve the problem in two steps.
1. Find a harmonic function $u_1(x, y)$, which satisfies the following boundary conditions

$$u_1(0, y) = \varphi_0(y), \quad u_1(a, y) = \varphi_1(y),$$

$$u_1(x, 0) = 0, \qquad u_1(x, b) = 0.$$

2. Determine a harmonic function $u_2(x, y)$, which satisfes the following boundary conditions

$$u_2(0, y) = 0, \qquad u_2(a, y) = 0,$$
$$u_2(x, 0) = \psi_0(x), \quad u_2(x, b) = \psi_1(x).$$

Now the function $u(x, y) = u_1(x, y) + u_2(x, y)$ is a solution of the given Dirichlet problem.

$$u(x, y) = B \frac{\mathfrak{Sin}\, \dfrac{\pi(b - y)}{a}}{\mathfrak{Sin}\, \dfrac{\pi b}{a}} \sin \frac{\pi x}{a} +$$

$$+ \frac{8Ab^2}{\pi^3} \sum_{n=0}^{\infty} \frac{\mathfrak{Sin}\, \dfrac{(2n + 1)\pi(a - x)}{b}}{(2n + 1)^3} \cdot \frac{\sin \dfrac{(2n + 1)\pi y}{b}}{\mathfrak{Sin}\, \dfrac{(2n + 1)\pi a}{b}}.$$

124. $u(x, y) = \dfrac{2A}{\pi} \sum\limits_{n=0}^{\infty} \dfrac{1}{n} e^{-(n\pi/a)y} \sin \dfrac{n\pi x}{a}.$

125. $u(r, \theta) = \dfrac{8A}{3} \mathfrak{Sin} \ln r \cdot \sin \theta.$

Hint: Introduce polar-coördinates.

126. $u(r, \theta) =$

$$= \frac{\alpha_0^{(2)} - \alpha_0^{(1)}}{\ln R_2 - \ln R_1} \ln r + \frac{\alpha_0^{(1)} \ln R_2 - \alpha_0^{(2)} \ln R_1}{\ln R_2 - \ln R_1} +$$

$$+ \sum_{n=1}^{\infty} \frac{(\alpha_n^{(1)} R_2^{-n} - \alpha_n^{(2)} R_1^{-n}) r^n - (\alpha_n^{(1)} R_2^n - \alpha_n^{(2)} R_1^n) r^{-n}}{R_1^n R_2^{-n} - R_1^{-n} R_2^n} \cos n\theta +$$

$$+ \frac{(\beta_n^{(1)} R_2^{-n} - \beta_n^{(2)} R_1^{-n}) r^n - (\beta_n^{(1)} R_2^n - \beta_n^{(2)} R_1^n) r^{-n}}{R_1^n R_2^{-n} - R_1^{-n} R_2^n} \sin n\theta,$$

with

$$\alpha_0^{(1)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(\theta)\, d\theta, \quad \alpha_n^{(1)} = \frac{1}{\pi} \int_{-\pi}^{\pi} f_1(\theta) \cos n\theta\, d\theta,$$

$$\beta_n^{(1)} = \frac{1}{\pi} \int_{-\pi}^{\pi} f_1(\theta) \sin n\theta \, d\theta;$$

$$\alpha_0^{(2)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_2(\theta) \, d\theta, \quad \alpha_n^{(2)} = \frac{1}{\pi} \int_{-\pi}^{\pi} f_2(\theta) \cos n\theta \, d\theta,$$

$$\beta_n^{(2)} = \frac{1}{\pi} \int_{-\pi}^{\pi} f_2(\theta) \sin n\theta \, d\theta.$$

Hint: Introduce polar-coördinates. Then the problem reduces to the solution of the equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta} = 0$$

with the conditions

$$u(r, \theta + 2\pi) = u(r, \theta),$$
$$u(R_1, \theta) = f_1(\theta), \quad u(R_2, \theta) = f_2(\theta),$$

where $f_1(\theta)$ and $f_2(\theta)$ are given functions, which can be expanded in a Fourier-series.
For a circle we obtain

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{R^2 - r^2}{R^2 - 2Rr \cos (t - \theta) + r^2} \, dt.$$

127. $u(r, \theta) = \alpha_0^{(2)} + \alpha_0^{(1)} R_1 \ln \dfrac{r}{R_2} +$

$$+ \sum_{k=1}^{\infty} \frac{(\alpha_k^{(1)} R_2^{-k} + k R_1^{-k-1} \alpha_k^{(2)}) r^k + (k R_1^{k-1} \alpha_k^{(2)} - R_2^k \alpha_k^{(1)}) r^{-k}}{k(R_1^{k-1} R_2^{-k} + R_2^k R_1^{-k-1})} \cos k\theta +$$

$$+ \sum_{n=1}^{\infty} \frac{(\beta_k^{(1)} R_2^{-k} + k R_1^{-k-1} \beta_k^{(2)}) r^k + (k R_1^{k-1} \beta_k^{(2)} - R_2^k \beta_k^{(1)}) r^{-k}}{k(R_1^{k-1} R_2^{-k} + R_2^k R_1^{-k-1})} \sin k\theta,$$

where $\alpha_0^{(1)}$, $\alpha_0^{(2)}$, $\alpha_k^{(1)}$, $\alpha_k^{(2)}$, $\beta_k^{(1)}$, $\beta_k^{(2)}$ have the same values as in problem **126.** (See also the hint for problem **126**).

128. $u(x, y) = A + \dfrac{A(b-2)}{2a}\, x -$

$$-\frac{4Ab}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cdot \frac{\mathfrak{Sin}\,\dfrac{(2k+1)\pi x}{b}}{\mathfrak{Sin}\,\dfrac{(2k+1)\pi a}{b}} \cos \frac{(2k+1)\pi y}{b}$$

129. $u(\rho, \varphi) = \dfrac{2A\alpha}{\pi} \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{\rho}{R}\right)^{n\pi/\alpha} \frac{\sin \dfrac{n\pi\varphi}{\alpha}}{n}.$

Hint: Introduce polar-coördinates.

130. $u(\rho, \theta) = \frac{1}{3}(1 - \rho^2) + \rho^2 \cos^2 \theta.$
Hint: Introduce spherical coördinates. Obviously the solution u is independent of φ. Therefore Laplace's equation becomes

$$\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) = 0.$$

131. $u(x, y) = \dfrac{Q}{2kab}\,[(y-b)^2 - (x-a)^2] + C,$

where C is a constant.
Hint: The problem is equivalent to Neumann's problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \frac{Q}{kb}, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = -\frac{Q}{ka}, \quad \left.\frac{\partial u}{\partial y}\right|_{y=b} = 0.$$

Here Q is the amount of heat which flows through OA into the plate and through OB out again: k is the coëfficient of heat conduction.

132. $u(x, y) = x(a - x) -$

$$-\frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{\mathfrak{Cof}\,\dfrac{(2n+1)\pi y}{a} \cdot \sin \dfrac{(2n+1)\pi x}{a}}{(2n+1)^3\, \mathfrak{Cof}\,\dfrac{(2n+1)\pi b}{2a}}.$$

Hint: One can look for a solution in the form of a sum $u = v + w$, where v is a solution of Poissons's equation which satisfies the conditions

$$v(0, y) = 0, \quad v(a, y) = 0$$

Put v in the form $Ax^2 + Bx + C$; w is a solution of Laplace's equation, which has as boundary values

$$w(0, y) = 0, \quad w(a, y) = 0,$$

$$w\left(x, -\frac{b}{2}\right) = -v(x), \quad w\left(x, \frac{b}{2}\right) = -v(x).$$

133. $u(x, y) = a^2 - (x^2 + y^2)$.

Hint: Introduce polar-coördinates.

134. $u(r, \theta) = -\frac{1}{24} r^3 \sin 2\theta +$

$$+ \frac{R^4}{48\pi} \int_{-\pi}^{\pi} \sin 2t \frac{R^2 - r^2}{R^2 - 2Rr\cos(t - \theta) + r^2}\, dt.$$

Hint: Look for a solution in the form of a sum $u = v + w$, where

$$v(x, y) = -\tfrac{1}{12} xy(x^2 + y^2)$$

is a special solution of Poisson's equation and $w(x, y)$ a solution of the equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0,$$

which satisfies the boundary conditions

$$w|_{r=R} = -v|_{r=R}.$$

135. $u(r, \varphi) = \Big[(a^4 + b^4)r^4 -$

$$- (a^6 + 2b^6)r^2 - (a^2 - 2b^2)\frac{a^4b^4}{r^2}\Big]\frac{\cos 2\varphi}{a^4 + b^4}.$$

Hint: Introduce polar-coördinates.

136. $u(r, z) =$

$$= \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{\mathfrak{Sin}\,\dfrac{\mu_n z}{R}}{\mathfrak{Sin}\,\dfrac{\mu_n h}{R}} \cdot \frac{J_0\left(\mu_n \dfrac{r}{R}\right)}{J_1^2(\mu_n)} \int_0^R \rho f(\rho) J_0\left(\mu_n \frac{\rho}{R}\right) d\rho,$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation $J_0(\mu) = 0$.
Hint: Integrate the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$

subject to the conditions

$$\begin{aligned} &u(0, z) \text{ is bounded} \\ &u(R, z) = 0; \\ &u(r, 0) = 0, \\ &u(r, h) = f(r). \end{aligned}$$

137. $$u(r, z) = \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{\mathfrak{Sin}\,\dfrac{\mu_n z}{R}}{\mathfrak{Sin}\,\dfrac{\mu_n h}{R}} \cdot \frac{J_0\left(\mu_n \dfrac{r}{R}\right)}{J_0^2(\mu_n)} \int_0^R \rho f(\rho) J_0\left(\mu_n \frac{\rho}{R}\right) d\rho,$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation $J_1(\mu) = 0$.
Hint: Replace the second boundary condition of problem **136** by the following:

$$\frac{\partial u(R, z)}{\partial r} = 0\cdot$$

138. $u(r, z) =$

$$= \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{\mathfrak{Sin}\,\dfrac{\mu_n z}{R}}{\mathfrak{Sin}\,\dfrac{\mu_n h}{R}} \cdot \frac{J_0\left(\dfrac{\mu_n r}{R}\right)}{\left(1 + \dfrac{a^2 R^2}{\mu_n^2}\right) J_0^2(\mu_n)} \cdot \int_0^R \rho f(\rho) J_0\left(\frac{\mu_n \rho}{R}\right) d\rho,$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of

$$\mu J_1(\mu) - \alpha R J_0(\mu) = 0.$$

Hint: Replace the second boundary condition of problem **136** by the following:

$$\frac{\partial u}{\partial r} + \alpha u|_{r=R} = 0.$$

139. $$u(r, z) = \frac{2}{h} \sum_{n=1}^{\infty} \sin \frac{n\pi z}{h} \cdot \frac{J_0\left(\dfrac{n\pi i r}{h}\right)}{J_0\left(\dfrac{n\pi i R}{h}\right)} \int_0^h f(x) \sin \frac{n\pi x}{h}\, dx.$$

140. $$u(r, z) = \frac{2}{h} \sum_{n=1}^{\infty} \cos \frac{n\pi z}{h} \cdot \frac{J_0\left(\dfrac{n\pi i r}{h}\right)}{J_0\left(\dfrac{n\pi i R}{h}\right)} \int_0^h f(x) \cos \frac{n\pi x}{h}\, dx.$$

Hint: Integrate the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$

subject to the conditions

$u(0, z)$ is bounded

$$\frac{\partial u(r, 0)}{\partial z} = \frac{\partial u(r, h)}{\partial z} = 0, \quad u(R, z) = f(z).$$

141. $$u = 4u_0 R\gamma \sum_{n=1}^{\infty} \frac{J_0\left(\dfrac{\mu_n r}{R}\right)}{J_0(\mu_n)(\mu_n^2 + R^2\gamma^2)} \times$$

$$\times \frac{\mu_n \operatorname{Cosh} \dfrac{\mu_n(H - z)}{R} + \beta R \operatorname{Sinh} \dfrac{\mu_n(H - z)}{R}}{\mu_n \operatorname{Cosh} \dfrac{\mu_n H}{R} + \beta R \operatorname{Sinh} \dfrac{\mu_n H}{R}},$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the following equation

$$\mu J_0'(\mu) + \gamma R J_0(\mu) = 0.$$

Hint: The problem can be reduced to the integration of Laplace's equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$

where the following conditions have to be satisfied.

$$u(0, z) \text{ is bounded,} \quad \left.\frac{\partial u}{\partial r} + \gamma u\right|_{r=R} = 0,$$

$$u(r, 0) = u_0, \quad \left.\frac{\partial u}{\partial z} + \beta u\right|_{z=H} = 0,$$

Here $\gamma = \frac{h_1}{k}$, $\beta = \frac{h_2}{k}$; h_1 and h_2 are the coëfficients of heat exchange of the lateral resp. upper and lower sides of the cylinder, k is the coëfficient of heat conduction.

142. $$u(r, \theta) = u_0 \sum_{n=1}^{\infty} (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)(4n+3)}{2\cdot 4\cdot 6\cdots(2n+2)} \times$$
$$\times \left(\frac{r}{R}\right)^{2n+1} P_{2n+1}(\cos\theta),$$

$$0 \leqslant \theta \leqslant \frac{\pi}{2};$$

The $P_n(x)$ are the Legendre polynomials.
Hint: The problem can be reduced to the integration of Laplace's equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\operatorname{ctg}\theta}{r^2}\frac{\partial u}{\partial \theta} = 0$$

subject to the following conditions:

$$u(R, \theta) = f(\theta) = \begin{cases} u_0 & \text{for} \quad 0 < \theta < \dfrac{\pi}{2}, \\ 0 & \text{for} \quad \theta = \dfrac{\pi}{2}. \end{cases}$$

143. $$v(r, \theta, \varphi) = \sum_{n=0}^{\infty} \frac{\psi_n(kr)}{\psi_n(kR)} Y_n(\theta, \varphi),$$

with

$$\psi_n(x) = \frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}},$$

$$Y_n(\theta, \varphi) = a_{n0}P_n(\cos\theta) + \\ + \sum_{m=1}^{n} (a_{nm}\cos m\varphi + b_{nm}\sin m\varphi)P_{n,m}(\cos\theta);$$

$$a_{nm} = \frac{2n+1}{2\delta_m\pi}\,\frac{(n-m)!}{(n+m)!}\int\int_{(S)} f(\theta, \varphi)P_{n,m}(\cos\theta)\cos m\varphi\, d\sigma;$$

$$b_{nm} = \frac{2n+1}{2\delta_m\pi}\,\frac{(n-m)!}{(n+m)!}\int\int_{(S)} f(\theta, \varphi)P_{n,m}(\cos\theta)\sin m\varphi\, d\sigma.$$

[$\delta_m = 2$ for $m = 0$ and $\delta_m = 1$ for $m > 0$; $P_{n,0}(x) = P_n(x)$].

It is presupposed, that k is not an eigenvalue of the following boundary value problem:

$$\Delta v + k^2 v = 0, \quad v|_{r=R} = 0.$$

Hint: Introduce spherical coördinates r, θ, φ. A solution is obtained in the form of a product.

$$v(r, \theta, \varphi) = R(r)Y(\theta, \varphi).$$

144. $$v(r, \theta, \varphi) = e^{im\varphi}\sum_{j=1}^{\infty} a_j\varphi_{l_j}(kr)\,\frac{P_{l_j,m}(\cos\theta)}{P_{l_j,m}(\cos\alpha)};$$

$$P_{l,m}(x) = (1-x^2)^{m/2}\frac{d^m P_l(x)}{dx^m}.$$

The $P_l(x)$ are the Legendre polynomials,

$$\varphi_{l_j}(kr) = \frac{Y_{l_j+\frac{1}{2}}(kb)J_{l_j+\frac{1}{2}}(kr) - J_{l_j+\frac{1}{2}}(kb)Y_{l_j+\frac{1}{2}}(kr)}{\sqrt{r}},$$

$l_1, l_2, l_3, \ldots$ are solutions of the equation

$$Y_{l+\frac{1}{2}}(kb)J_{l+\frac{1}{2}}(ka) - J_{l+\frac{1}{2}}(kb)Y_{l+\frac{1}{2}}(ka) = 0, \qquad (*)$$

$$a_j = \frac{\int_a^b f(r)\varphi_{l_j}^2(kr)\,dr}{\int_a^b \varphi_{l_j}^2(kr)\,dr}.$$

We have to assume that the equation (*) has for given k solutions l_j such that $P_{l_j,m}(\cos\alpha) \neq 0$ for all j.

If a solution l_{j_0} satisfies

$$P_{l_{j_0},m}(\cos\alpha) = 0$$

then it is necessary and sufficient in order that the problem can be solved that

$$a_{j_0} = 0, \text{ that is } \int_a^b f(r)\varphi_{l_{j_0}}(kr)\,dr = 0.$$

If it is possible to satisfy this condition, the solution can be obtained in the form of a series in which the term corresponding to $j = j_0$ is left out. In this case the solution is not unique. Terms of the form

$$A_{j_0}\varphi_{l_{j_0}}(kr)P_{l_{j_0},m}(\cos\theta)e^{im\varphi}$$

can be added. A_{j_0} is an arbitrary constant.

Hint: Introduce spherical coördinates r, θ, φ. and assume

$$v(r, \theta, \varphi) = R(r)P(\theta)\,e^{im\varphi}.$$